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QUALITATIVE PROPERTIES OF SOLUTIONS OF NON-UNIFORMLY AND STRONGLY DEGENERATE SECOND-ORDER ELLIPTIC-PARABOLIC EQUATIONS

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Abstract. In this paper we consider a class of second order elliptic-parabolic equations of non-divergent structure, allowing for non-uniform degeneracy in a finite number of points of the region. For the strong solutions of the indicated equations we prove an internal a priori estimate of the Holder norm, and for positive solutions - Harnack type inequalities.

Keywords: Elliptic-parabolic equation, unequally and degenerate, Sobolev space.

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1. Introduction

This paper is devoted to the study of qualitative properties of solutions of non-uniformly degenerate non-divergent second-order elliptic-parabolic equations with discontinuous coefficients. At first, we prove the theorem on the increase of positive solutions of the above equations, after which an internal a priori estimate of the Holder norm is established for them. Next, following the method of operation [11], for non-negative solutions of the above equations we prove the analog of the classical Harnack's inequality. We will limit ourselves here to mentioning the works of [1]- [11].

2. Theorem on increasing positive solutions

Let E_n and R_{n+1}-euclidean point spaces $x = (x_1, \dots, x_n)$ and $(x,t) = (x_1, \dots, x_n, t)$ respectively.

Let us consider in the bounded domain $D \subset R_{n+1}$ (not necessarily cylindrical), $n \geq 1$, $(0,0) \in D$ the following elliptic-parabolic equation

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x,t)u_{ij} + \varphi(0-t)u_{tt} - u_t = 0, (x,t) \in D. \qquad (1)$$

Suppose that $\|a_{ij}(x,t)\|$ is a real symmetric matrix $\xi \in E_n$ and $(x,t) \in aD$ condition fulfilled (see, [3])

$$\inf_D \sum_{i=1}^{n} \frac{a_{ii}(x,t)}{\lambda_i(x,t)} = \gamma, \tag{2}$$

$$\sigma = \sup_D \sum_{i=1}^{n} \frac{a_{ii}(x,t)}{\lambda_i(x,t)} \left[\sum_{i,j=1}^{n} \frac{a_{ij}^2(x,t)}{\lambda_i(x,t)\lambda_j(x,t)} \Big/ \left(\sum_{i=1}^{n} \frac{a_{ii}(x,t)}{\lambda_i(x,t)} \right)^2 \right] -$$

$$- \frac{1}{n-e^2} < 0 \tag{3}$$

$$\varphi(z) \in C^1[-T,0], \varphi(z) \geq 0, \varphi'(z) \geq 0, \varphi(0) = 0,$$

$$\varphi'(0) = 0, \varphi(z) \geq \beta_1 z\varphi'(z), \beta_1 > 0 - \text{constant}. \tag{4}$$

Here $u = u(x,t), u_t = \frac{\partial u}{\partial t}, u_{tt} = \frac{\partial^2 u}{\partial t^2}, u_i = \frac{\partial x}{\partial x_i}, u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}, i,j = 1,\dots,n,$ $\gamma \in (0,1]$- constant,

$$e = \frac{\inf_D \sum_{i=1}^{n} \frac{a_{ii}(x,t)}{\lambda_i(x,t)}}{\sup_D \sum_{i=1}^{n} \frac{a_{ii}(x,t)}{\lambda_i(x,t)}},$$

$$\lambda_i(x,t) = \left[\frac{\omega_i^{-1}\left(\rho(x)+\sqrt{|t|}\right)}{\rho(x)+\sqrt{|t|}} \right]^2, i = 1,\dots,n,$$

where

$$\rho(x,t) = \sum_{i=1}^{n} \omega_i(|x_i|)$$

Concerning the functions $\omega_i(z)$ $(i = 1,\dots,n)$ will suppose satisfying the following conditions: $\omega_i(t)$ –continuous and strongly monotonically increasing on $[0, \mathrm{diam}D]$ function, $\omega_i(0) = 0, (i = 1,\dots,n)$, $\omega_i^{-1}(z), (i = 1,\dots,n)$ function, the inverses of $\omega_i(z)$, and moreover. (see, [2])

$$\alpha \cdot \omega_i(R) \leq \omega_i(\eta R) \leq \beta \omega_i(R), \qquad R \in \left(0, \frac{\mathrm{diam}\, D}{2}\right], \tag{5}$$

$$\left(\frac{\omega_i^{-1}(R)}{R} \right)^{q-1} \cdot \int_0^{\omega_i^{-1}(R)} \left(\frac{\omega_i(\tau)}{\tau} \right)^q d\tau \leq A \cdot R,$$

$$R \in (0, \mathrm{diam}D), i = 1,\dots,n, \tag{6}$$

for some $\alpha, \beta \in (1,\infty), \eta > 0$ and $q > n$ and constants A and α, β do not depend on R. Moreover, suppose that the elements of the matrix $\|a_{ij}(x,t)\|$ are measurable functions in D. The condition (3) is called a parabolic condition of Cordes type [4], [1]-[3].

By the solution of the equation (1) we will understand its classical solution, i.e., the function $u \in C^{2,2}(D) \cap C(\overline{D})$, converting the equality (1) to the identity. The function $u \in C^{2,2}(D)$ is called L- subelliptic-parabolic in D, if $Lu \geq 0$ for $(x,t) \in D$.The function $u(x,t)$ is called L-superelliptic-parabolic in D, if the function-$u(x,t)$ is L-subelliptic-parabolic in D.

Let $x^0 \in E_n, R > 0, k > 0$, denote by $E_R^{x^0}(k)$ the ellipsoid

$$\left\{x: \sum_{i=1}^{n} \frac{\left(x_i - x_i^0\right)^2}{\left(\omega_i^{-1}(R)\right)^2} < k^2\right\},$$

through $\Pi_{R:k}(x^0)$ –parallelepiped $\{x: \left|x_i - x_i^0\right| < k \cdot \omega_i^{-1}(R), i = 1,\ldots,n\}$ and for $t^1 < t^2$, through $C_{R:k}^{t^1,t^2}(x^0)$ is the cylinder $E_R^{x^0}(k) \times (t^1, t^2)$.

Let

$$C^1 = C_{R:17}^{-\frac{9bR^2}{8},0}(0), C^2 = C_{R:1}^{-\frac{bR^2}{16},0}(0), C^3 = C^1 \backslash \bar{C}^2, C^4 = C_{R:9}^{-\frac{bR^2}{8},0}(0),$$

$$(x^0, t^0) \in \Gamma(C^4), C^5 = C_{R:8}^{t^0 - bR^2, t^0}(x^0) = C^5(x^0, t^0),$$

$$C^6 = C^6(x^0, t^0) = C_{R:1}^{t^0 - \frac{bR^2}{4}, t^0}(x^0), C^7 = C^7(x^0, t^0) = C_{R:1}^{t^0 - bR^2, t^0 - \frac{bR^2}{4}}(x^0),$$

where the constant $b \in (0,1]$ will be determined later. Through $S(C)$ will denote the lateral surface of the cylinder C,and through $F(C)$ is its lowest base.

For $S > 0$ and $\beta > 0$ we introduce the following function

$$G_R^{S,\beta}(x,t) = \begin{cases} t^{-S} \cdot \exp\left[-\frac{1}{4\beta t} \sum_{i=1}^{n} \frac{x_i^2}{\left(\omega_i^{-1}(R)\right)^2}\right], & t > 0, \\ 0 & t \leq 0. \end{cases}$$

Lemma 2.1. If with respect to the coefficients of the operator L conditions (2) - (4) are satisfied, there exist $S(\gamma, n, b)$ and $\beta(\gamma, n, b)$ such that for any fixed point $(y, \tau) \in C^3$, the function $G_R^{S,\beta}(x - y, t - \tau)$ is L-subelliptic-parabolic in $C^3 \backslash \{(y, \tau)\}$ at $R \leq 1$.

Proof. It is enough to consider the case $> \tau$. For the prostate, we will denote the function $G_R^{S,\beta}(x,t)$ by $G(x,t)$. We have

$$J = \frac{LG\,(x - y, t - \tau)}{G\,(x - y, t - \tau)} \cdot (t - \tau) =$$

$$= \frac{1}{4\beta^2(t - \tau)} \cdot \sum_{i,j=1}^{n} a_{ij}(x,t) \cdot \frac{(x_i - y_i)(x_j - y_j)}{\left(\omega_i^{-1}(R)\right)^2 \cdot \left(\omega_j^{-1}(R)\right)^2} -$$

$$- \frac{1}{2\beta} \cdot \sum_{i=1}^{n} \frac{a_{ii}(x,t)}{\left(\omega_i^{-1}(R)\right)^2} + S - \frac{1}{4\beta(t - \tau)} \sum_{i=1}^{n} \frac{(x_i - y_i)^2}{\left(\omega_i^{-1}(R)\right)^2} -$$

$$-\varphi(0 - t)\frac{S(S+1)}{t - \tau} - \frac{(S+1) \cdot \varphi(0 - t)}{2\beta(t - \tau)^2} \cdot \sum_{i=1}^{n} \frac{(x_i - y_i)^2}{\left(\omega_i^{-1}(R)\right)^2} +$$

$$+\frac{\varphi(0-t)}{16\beta^2\cdot(t-\tau)^3}\cdot\left[\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{\left(\omega_i^{-1}(R)\right)^2}\right]^2\geq$$

$$\geq\frac{1}{4\beta^2(t-\tau)}\cdot\sum_{i=1}^{n}\frac{a_{ii}(x,t)}{\lambda_i(x,t)}\cdot\sum_{i=1}^{n}\lambda_i(x,t)\cdot\frac{(x_i-y_i)^2}{\left(\omega_i^{-1}(R)\right)^4}-$$

$$-\frac{1}{2\beta}\cdot\sum_{i=1}^{n}\frac{a_{ii}(x,t)}{\lambda_i(x,t)}\cdot\sum_{i=1}^{n}\lambda_i(x,t)\cdot\frac{1}{\left(\omega_i^{-1}(R)\right)^2}+$$

$$+S-\frac{1}{4\beta(t-\tau)}\cdot\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{\left(\omega_i^{-1}(R)\right)^2}. \tag{7}$$

If $(x,t)\in C^3$, then $|x_i|\leq 17\cdot\omega_i^{-1}(R), i=1,\dots,n$.
Therefore

$$\omega_i(|x_i|)\leq\omega_i\left(17\omega_i^{-1}(R)\right)\leq\beta R,$$

$$\rho(x)=\sum_{i=1}^{n}\omega_i(|x_i|)\leq\beta nR,\ \ \sqrt{|t|}\leq\sqrt{\frac{9b}{8}}\leq 2R,$$

$$\rho(x)+\sqrt{|t|}\leq(\beta n+2)R. \tag{8}$$

On the other hand, for $(x,t)\in C^3$ either $\sum_{i,j=1}^{n}\frac{x_i^2}{\left(\omega_i^{-1}(R)\right)^2}\geq 1$, either $|t|\geq\frac{bR^2}{16}$. Thus, either there exists a natural number $i_0, 1\leq i_0\leq n$ such that $\left|x_{i_0}\right|\geq\frac{1}{\sqrt{n}}\omega_{i_0}^{-1}(R)$, either $|t|\geq\frac{\sqrt{b}}{4}R$.

Conclude that

$$c_{1.1}(n,b)\left(\frac{\omega_i^{-1}(R)}{R}\right)^2\leq\lambda_i(x,t)\leq c_{1.2}(n,b)\left(\frac{\omega_i^{-1}(R)}{R}\right)^2, i=1,\dots,n.$$

Furthermore

$$\left(\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2}\leq\left(\sum_{i=1}^{n}\frac{x_i^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2}+$$

$$+\left(\sum_{i=1}^{n}\frac{y_i^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2}\leq 17+17=34.$$

We get

$$J \geq \frac{\gamma}{4\beta^2(t-\tau)} \cdot \frac{c_{1.1}}{R^2} \cdot \sum_{i=1}^{n} \frac{(x_i-y_i)^2}{\left(\omega_i^{-1}(R)\right)^2} -$$

$$-\frac{n}{\gamma \cdot 2\beta} \cdot \frac{c_{1.2}}{R^2} + S - \frac{1}{4\beta(t-\tau)} \cdot \sum_{i=1}^{n} \frac{(x_i-y_i)^2}{\left(\omega_i^{-1}(R)\right)^2} \geq$$

$$\geq \frac{1}{4\beta(t-\tau)} \cdot \left(\frac{\gamma \cdot c_{1.1}}{\beta} - 1\right) \sum_{i=1}^{n} \frac{(x_i-y_i)^2}{\left(\omega_i^{-1}(R)\right)^2} + S - \frac{nc_{1.2}}{2\gamma\beta}. \tag{9}$$

Therefore, assuming in (9)

$$\beta = \gamma c_{1.1} \tag{10}$$

in conclusion

$$J \geq S - \frac{n \cdot c_{1.2}}{2\gamma^2 \cdot c_{1.1}}. \tag{11}$$

Now it's enough to choose

$$S = \frac{n \cdot c_{1.2}}{2\gamma^2 \cdot c_{1.1}} \tag{12}$$

and the lemma is proven.

All further on, we will assume that the parameters β and S functions $G_R^{S,\beta}(x,t)$ are chosen in accordance with the equations (10) and (11). Choose and fix an arbitrary $b \in (0,1)$ satisfying the inequality

$$b\beta S \leq \frac{49}{4}. \tag{13}$$

Let $E - B$-set, situated in C^3. Name the measure μ on $E, (S, \beta, R)$ as admissible, if

$$\int_E G_R^{S,\beta}(x-y, t-\tau)\, d\mu(y,\tau) \leq 1 \quad \text{for} \quad (x,t) \notin E.$$

Number $P_R^{S,\beta}(E) = \sup \mu(E)$, where the exact upper bound is taken over all (S, β, R)-admissible measures, is called the subelliptic-parabolic (S, β, R) capacity of the set E.

In the following, to shorten the notation, we will denote $P_R^{S,\beta}(E)$, simply by $P_R(E)$.

Lemma 2.2. Let $B = C_{R:\rho}^{t^0-\rho^2R^2, t^0}, B \subset C^3$, where $\rho > 0, R \in (0,1]$, then

$$c_{1.3}(S,\beta) \cdot (\rho R)^{2S} \leq P_R(B) \leq c_{1.4}(S,\beta) \cdot (\rho R)^{2S}.$$

Proof. Let's consider the function

$$W(x,t) = G(x,t; x^0, t^0 - \rho^2R^2)\ , (x,t) \in \bar{B}.$$

If $\leq t^0 - \rho^2R^2$, then $W(x,t) = 0$. If $x \,\overline{\in}\, E_R^{x^0}(\rho)$, then

$$W(x,t) = (t - t^0 + \rho^2R^2)^{-S} \times$$

$$\times \exp\left[-\frac{1}{4\beta(t-t^0+\rho^2R^2)}\cdot\sum_{i=1}^{n}\frac{\left(x_i-x_i^0\right)^2}{\left(\omega_i^{-1}(R)\right)^2}\right]\le$$

$$\le (t-t^0+\rho^2R^2)^{-S}\cdot\exp\left(-\frac{\rho^2}{4\beta(t-t^0+\rho^2R^2)}\right).$$

Consider the auxiliary function

$$Z(v)=v^{-S}\cdot\exp\left(-\frac{\rho^2}{4\beta v}\right),\quad v>0.$$

Solving the equations $Z`(v)=0$, we find that $v=\frac{\rho^2}{4\beta S}$. Thus the function $W(x,t)$ reaches a maximum at $t=\frac{\rho^2}{4\beta S}+t^0-\rho^2R^2$. There by for $x \overline{\in} E_R^{x^0}(\rho)$

$$W(x,t)=\left(\frac{\rho^2}{4\beta S}\right)^{-S}\cdot\exp(-S)=(4\beta S)^S\cdot(\rho R)^{-2S}\cdot R^{2S}\cdot\exp(-S).$$

Now, let's say $\ge t^0$. Since at $\rho>0, t-t^0+\rho^2R^2\ge\rho^2R^2$, $(t-t^0+\rho^2R^2)^{-S}\le(\rho^2R^2)^{-S}$ and

$$\exp\left[-\frac{1}{4\beta(t-t^0+\rho^2R^2)}\cdot\sum_{i=1}^{n}\frac{\left(x_i-x_i^0\right)^2}{\left(\omega_i^{-1}(R)\right)^2}\right]\le 1,$$

then

$$W(x,t)\le(\rho R)^{-2S}. \tag{14}$$

Combining (13) and (14), we conclude that

$$\sup_{R_{n+1}\backslash\bar{B}} W(x,t)\le a\cdot(\rho R)^{-2S}, \tag{15}$$

where $a=\max=\{1;(4\beta S)^S\cdot e^{-S}\cdot R^{2S}\}$.

Now consider the singular measure μ, concentrated at the point $(x^0,t^0-\rho^2R^2)$ with density $\frac{1}{a}\cdot(\rho R)^{2S}$.

We have for $(x,t)\overline{\in} B$ taking into account (15)

$$\int_B G(x,y;t,\tau)d\mu(y,\tau)=\int_{\{x^0,t^0-\rho^2R^2\}} W(x,y)d\mu\{x^0,t^0-\rho^2R^2\}\le$$

$$\le a\cdot(\rho R)^{-2S}\cdot\mu\{(x^0,t^0-\rho^2R^2)\}=1$$

that is, the measure μ is (S,β,R) admissible. Therefore $P_R(B)=\sup\mu(B)$ and $P_R(B)\ge\mu(B)=\mu\{(x^0,t^0-\rho^2R^2)\}=\frac{1}{a}\cdot(\rho R)^{2S}$, which coincides with $P_R(B)\ge c_{1.3}(\rho R)^{2S}$ at $c_{1.3}=\frac{1}{a}$.

The proof of the lemma is complete.

Lemma 2.3. Let C^5 contain a region D having limit points at $\Gamma(C^5)$ and intersecting C^6. Let D further define a positive L-subelliptic-parabolic function

$u(x,t)$, continuous in $\overline{D}$ and going to zero on $\Gamma(D) \cap C^5$. Then if $E_R = C^7 \backslash D$ and $R \leq 1$, then

$$\sup_D \geq \left(1 + \eta_1(\gamma, n) \cdot R^{-2S} \cdot P_R(E_R)\right) \cdot \sup_{D \cap C^6} u. \tag{16}$$

Proof. Without losing generality, we can consider that $P_R(E_R) > 0$, otherwise inequality (16) is obvious. Let us fix an arbitrary $0 < \varepsilon < P_R(E_R)$ and let μ a measure of E_R is such that

$$U(x,t) = \int_B G(x-y, t-\tau) d\mu(y,\tau) \leq 1, \ \ (x,t) \overline{\in} E_R. \tag{17}$$

$$\mu(E_R) > P_R(E_R) - \varepsilon, \tag{18}$$

where $G(x,t) = G_R^{S,\beta}(x,t)$. Let us fix a point $(y,\tau) \in C^7$ and an n-dimensional vector $x \in \partial E_R^{x^0}(8)$. Let's find that value now $t > \tau$ at which the function $v(t) = G(x-y, t-\tau)$ reaches its maximum. Setting the derivative v_t equal to zero, we obtain

$$t - \tau = \frac{1}{4\beta S} \cdot \sum_{i=1}^{n} \frac{(x_i - y_i)^2}{\left(\omega_i^{-1}(R)\right)^2}.$$

On the other hand

$$\left(\sum_{i=1}^{n} \frac{(x_i - y_i)^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{\frac{1}{2}} \geq \left(\sum_{i=1}^{n} \frac{\left(x_i - x_i^0\right)^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{\frac{1}{2}} -$$

$$\left(\sum_{i=1}^{n} \frac{\left(y_i - x_i^0\right)^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{\frac{1}{2}} \geq 8 - 1 = 7.$$

Therefore

$$t - \tau \geq \frac{1}{4\beta S} \cdot 49.$$

Then

$$bR^2 \leq \frac{49}{4\beta S} \leq t - \tau.$$

Given the monotonicity of $v(t)$ up to the first maximum, we conclude

$$\sup_{\substack{(x,t) \in S(C^5) \\ (y,\tau) \in C^7}} G(x-y, t-\tau) \leq R^{-2S} \cdot b^S \cdot e^{-S}. \tag{19}$$

Now let's evaluate $\inf\limits_{\substack{(x,t) \in S(C^6) \\ (y,\tau) \in C^7}} G(x-y, t-\tau)$. Let $x \in \partial E_R^{x^0}(1)$, $y \in \partial E_R^{x^0}(1)$. Then

$$\left(\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{\frac{1}{2}} \geq \left(\sum_{i=1}^{n}\frac{\left(x_i-x_i^0\right)^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{\frac{1}{2}} +$$
$$+\left(\sum_{i=1}^{n}\frac{\left(y_i-x_i^0\right)^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{\frac{1}{2}} \leq 1+1\leq 2.$$

On the other hand outside $\frac{bR^2}{4} < t-\tau < bR^2$. Thus, we obtain

$$\inf_{\substack{(x,t)\in S(C^6)\\ (y,\tau)\in C^7}} G(x-y,t-\tau) \geq R^{-2S}\cdot e^{-S}\cdot e^{-\frac{16S}{49}}. \tag{20}$$

Let us consider the auxiliary function

$$W(x,t)=M\big(1-U(x,T)+R^{-2S}\cdot(be)^{-S}\cdot P_R(E_R)\big)-u(x,t),$$

where $M=\sup\limits_{D} u$.

Clearly, the function $W(x,t)$ is L-superelliptic-parabolic in D by virtue of Lemma 2.1.

According to the inequality (19) $W(x,t)\geq 0$ for $(x,t)\in \Gamma(D)\cap S(C^5)$. Moreover $W(x,t)\geq 0$ for $(x,t)\in\Gamma(D)\cap C^5$ by virtue of the inequality (17). Finally $W(x,t)\geq 0$ for $(x,t)\in F(C^5)$, E_R, for there $U(x,t)=0$. Thus $W(x,t)\geq 0$ for $(x,t)\in\Gamma(D)$. By the maximum principle $W(x,t)\geq 0$ in D and, in particular, given (20) and (18)

$$\sup_{D\cap C^6} \leq M\left(1-\inf_{C^6} U(x,y)+R^{-2S}\cdot(be)^{-S}\cdot P_R(E_R)\right)\leq$$
$$\leq M\left(1-(b)^{-S}\cdot\exp\left(-\frac{16S}{49}\right)\cdot(P_R(E_R)-\varepsilon)R^{-2S}+\right.$$
$$\left.+R^{-2S}(b)^{-S}\cdot e^{-S}\cdot P_R(E_R)\right)=M\left[1-(b)^{-S}\cdot\left(\exp\left(-\frac{16S}{49}\right)-\right.\right.$$
$$\left.\left.-\exp(-S)\right)R^{-2S}\cdot P_R(E_R)+\varepsilon\cdot(b)^{-S}\cdot\exp\left(-\frac{16S}{49}\right)R^{-2S}\right].$$

Considering now the arbitrariness of ε and the notation $b^{-S}\left(\exp\left(-\frac{16S}{49}\right)-\exp(-S)\right)$ by η_1, we obtain

$$\sup_{D\cap C^6} u\leq M\big(1-\eta_1\cdot R^{-2S}\cdot P_R(E_R)\big).$$

Hence the necessary estimate of (16) follows.

Lemma 2.3 is proved.

Corollary 2.1. If the conditions of Lemma 2.3 are satisfied and E_R contains the cylinder $C_{R:\rho}^{t`-\rho^2R^2,t`}(x`)$, then

$$\sup_{D} u\geq\big(1+\eta_2(\gamma,n,\rho)\big)\cdot\sup_{D\cap C^6} u.$$

To prove it is enough to apply Lemma 2.2.

Lemma 2.4. Let C^5 is located in the area D having limit points on $\Gamma(C^5)$ and intersecting C^6. Suppose further that in D there is a positive L-subellipticparabolic function $u(x,t)$ continuous in $\overline{D}$ and going to zero on $\Gamma(D) \cap C^5$. Then there exists $\delta = \delta(\gamma, n)$ such that if $E_R = C^7 \backslash D$ and mes$D \le \delta \cdot$ mesC^5, then

$$\sup_D u \ge (1+\eta_3) \sup_{D\cap C^6} u, \tag{21}$$

where $\eta_3 = \frac{b}{2}$.

Proof. Consider the auxiliary function

$$W_1(x,t) = M\left(1 + \frac{b}{64} \cdot \sum_{i=1}^{n} \frac{\left(x_i - x_i^0\right)^2}{\left(\omega_i^{-1}(R)\right)^2} + \frac{t^0 - t}{R^2} - b\right) - u(x,t),$$

where M- has the same point as in the preceding lemma.

We have

$$W_1|_{S(C^5)} \ge M\left(1 + \frac{b}{64} \cdot 64 - b\right) - M = 0,$$
$$W_1|_{F(C^5)} \ge M\left(1 + \frac{bR^2}{R^2} - b\right) - M = 0,$$
$$W_1|_{\Gamma(D)\cap(C^5)} \ge M(1-b) \ge 0.$$

Thus

$$W_1|_{\Gamma(D)} \ge 0. \tag{22}$$

On the other hand

$$\sup_{D\cap C^6} W_1(x,t) \le M \cdot \sup_{D\cap C^6}\left(1 + \frac{b}{64} \cdot \sum_{i=1}^{n} \frac{\left(x_i - x_i^0\right)^2}{\left(\omega_i^{-1}(R)\right)^2} + \frac{t^0 - t}{R^2} - b\right) \le$$
$$\le \left(1 + \frac{b}{64} \cdot 1 + \frac{bR^2}{4R^2} - b\right) = M\left(1 - \frac{47b}{64}\right). \tag{23}$$

Moreover

$$LW_1 = \frac{bM}{32} \cdot \sum_{i=1}^{n} \frac{a_{ii}(x,t)}{\left(\omega_i^{-1}(R)\right)^2} + \frac{M}{R^2} - Lu \le$$
$$\le \frac{bM}{32} \cdot \left(\sum_{i=1}^{n} \frac{a_{ii}^2(x,t)}{\lambda_i^2(x,t)}\right)^{1/2} \cdot \left(\sum_{i=1}^{n} \lambda_i^2(x,t) \cdot \frac{1}{\left(\omega_i^{-1}(R)\right)^4}\right)^{1/2} + \frac{M}{R^2} \le$$
$$\le \frac{bM}{32} \cdot \sum_{i=1}^{n} \frac{a_{ii}\,(x,t)}{\lambda_i\;(x,t)} \cdot \sum_{i=1}^{n} \frac{\lambda_i\;(x,t)}{\left(\omega_i^{-1}(R)\right)^2} + \frac{M}{R^2} \le \frac{bM}{32} \cdot c_{2.1.2} \cdot \frac{n}{R^2} + \frac{M}{R^2} =$$
$$= \left(1 + \frac{bn \cdot c_{2.1.2}}{32\gamma}\right) \cdot \frac{M}{R^2} = c_{2.1.4}(\gamma, n, b) \cdot \frac{M}{R^2}. \tag{24}$$

Note that without loss of generality, we can assume that the coefficients of the operator L are indefinitely differentiable in $\overline{D}$. Let D' be the intersection of some neighborhood of the region D with the cylinder $\tilde{C}^5 = C_{R:8\frac{1}{2}}^{t^0-bR^2,t^0}(x^0)$ such that

$$\text{mes } D` \le 2 \text{ mes } D. \tag{25}$$

Let's introduce the function $\zeta(x,t)$ as follows: $\zeta(x,t) = 1$ when $(x,t) \in D, \zeta(x,t) = 0$ when $(x,t) \overline{\in} D', 0 \le \zeta(x,t) \le 1$, $\zeta(x,t)$ is indefinitely differentiable in the closure of $\tilde{C}^5$.

Let $z(x,t)$ be the classical solution of the first boundary value problem

$$\begin{cases} Lz = -\dfrac{c_{2.1.4}`}{R^2} \cdot \zeta(x,t), \quad (x,t) \in \tilde{C}^5; \\ z \quad |_{\Gamma(\tilde{C}^5)} = 0. \end{cases} \tag{26}$$

It follows from the maximum principle that $z(x,t) \ge 0$ for $(x,t) \in \tilde{C}^5$. Moreover, according to an inequality of type A. D. Alexandrov type inequality for second order elliptic-parabolic operators (see [7])

$$\sup_{\tilde{C}^5} z(x,t) \le \frac{c_{1.5}(n)\left[\text{mes } E_R^{x^0}(8{,}5)\right]^{\frac{1}{n+1}}}{\left(\inf\limits_{\tilde{C}^5} \det\left(\left\|a_{ij}(x,t)\right\|\right)\right)^{\frac{1}{n+1}}} \cdot \left\| \frac{\zeta \cdot c`_{1.5}}{R^2} \right\|_{L_{n+1}(\tilde{C}^5)}. \tag{27}$$

By proceeding in the same way as in the proof of (8), we obtain for $(x,t) \in \tilde{C}^5$

$$\lambda_i(x,t) \ge c_{2.1.1} \cdot \left(\frac{\omega_i^{-1}(R)}{R} \right)^2, \qquad i = 1, \dots, n. \tag{28}$$

It follows that

$$\begin{aligned} \inf_{\tilde{C}^5} \det\left(\left\|a_{ij}\right\|\right) &\ge \gamma^n \cdot \inf_{\tilde{C}^5} \prod_{i=1}^{n} \lambda_i(x,t) \ge \\ &\ge (\gamma \cdot c_{1.1})^n \cdot R^{-2n} \cdot \prod_{i=1}^{n} \left(\omega_i^{-1}(R)\right)^2. \end{aligned} \tag{29}$$

From (27) - 29), taking into account (25), we conclude that

$$\begin{aligned} \sup_{\tilde{C}^5} z(x,t) &\le \frac{c_{1.6}(\gamma,n)\left[\prod_{i=1}^{n} \omega_i^{-1}(R)\right]^{\frac{1}{n+1}}}{R^{-\frac{2n}{n+1}} \cdot \left[\prod_{i=1}^{n} \omega_i^{-1}(R)\right]^{\frac{2}{n+1}}} \times \frac{1}{R^2}\left(\text{mes } \tilde{C}^5\right)^{\frac{1}{n+1}} \le \\ &\le c_{1.6} \cdot R^{-\frac{2}{n+1}} \cdot \left[\prod_{i=1}^{n} \omega_i^{-1}(R)\right]^{-\frac{1}{n+1}} \cdot (\text{mes } D`)^{\frac{1}{n+1}} \le \\ &\le c_{1.6} \cdot R^{-\frac{2}{n+1}} \cdot \left[\prod_{i=1}^{n} \omega_i^{-1}(R)\right]^{-\frac{1}{n+1}} \cdot (2\text{mes } D)^{\frac{1}{n+1}} \le \end{aligned}$$

$$\leq c_{1.6} \cdot R^{-\frac{2}{n+1}} \cdot \left[\prod_{i=1}^{n} \omega_i^{-1}(R)\right]^{-\frac{1}{n+1}} \cdot 2^{\frac{1}{n+1}} \times$$
$$\times\, \delta^{\frac{1}{n+1}} \cdot (\mathrm{mes} C^5\,)^{\frac{1}{n+1}} \leq c_{1.6} \cdot (2b\delta\,)^{\frac{1}{n+1}}. \tag{30}$$

Let's consider one more auxiliary function

$$W_2(x,t) = W_1(x,t) + M \cdot z\,(x,t).$$

It follows from (24) and (25) that $LW_2 \leq 0$ for $(x,t) \in D$. From (22) and the non-negativity of the function $z(x,t)$, we obtain that $W_2|_{\Gamma(D)} \geq 0$. By the maximum principle, we conclude that $W_2(x,t) \geq 0$ in D, in particular, given (23) and (30)

$$\sup_{D\cap C^6} u(x,t) \leq M\left[1 - \frac{47b}{64} + 2^{\frac{1}{n+1}} \cdot b^{\frac{1}{n+1}} \cdot \delta^{\frac{1}{n+1}} \cdot c_{1.6}\right]. \tag{31}$$

We choose and fix δ so small that

$$2^{\frac{1}{n+1}} \cdot c_{1.6} \cdot b^{\frac{1}{n+1}} \cdot \delta^{\frac{1}{n+1}} \leq \frac{15b}{64}.$$

Then the necessary estimate of (31) follows from (21).

The lemma is proved.

Corollary 2.2. Let the conditions of Corollary 2.1 be satisfied with respect to the region D and let D define a positive L superelliptic-parabolic function $v(x,t)$ continuous in $\overline{D}$ and turning to unity on $\Gamma(D) \cap C^5$.

Then, if $R \leq R_0$, then

$$\inf_{D\cap C^6} v \geq \eta_2`` = \frac{\eta_2`}{1+\eta_2`}, \tag{32}$$

where $\eta_2` = \frac{\eta_2}{2}, R_0 = R_0(n,\gamma)$.

Proof. Let $D' = \{(x,t) \quad | \, (x,t) \in D, v(x,t) < 1\}$. Consider the function $u(x,t) = 1 - v(x,t)$ for $(x,t) \in D'$. Applying Corollary 2.1 to the function $u(x,t)$ we get

$$1 - \inf_{D`} v \geq (1+\eta_2`)\left(1 - \inf_{D`\cap C^6} v\right),$$

i.e.

$$\inf_{D`\cap C^6} v \geq \frac{\eta_2`}{1+\eta_2`}. \tag{33}$$

Now it is enough to remark that the required estimate of (32) is also proved.

Analogously it is proved

Corollary 2.3. Let the conditions of Lemma 2.4 be satisfied with respect to the region D. Lemma 2.4 and there is defined in D a positive L-superalleptic-parabolic function $v(x,t)$, continuous in $\overline{D}$ and turning to one on $\Gamma(D) \cap C^5$. Then if $R \leq R_0$, then.

$$\inf_{D\cap C^6} v \geq \eta_3`` = \frac{\eta_3`}{1+\eta_3`}, \tag{34}$$

where $\eta_3` = \frac{\eta_3}{2}, R_0 = R_0(n, \gamma)$.

Lemma 2.5. Let C^5 be a domain D which has limit points on $\Gamma(C^5)$ and intersecting C^6. Let D further define a positive L-supereliptic-parabolic function $v(x,t)$, continuous in $\overline{D}$ and converging to unity on $\Gamma(D) \cap C^5$. Then if $E_R = C^7 \setminus D$, $\text{mes} E_R \geq \sigma_1 \cdot \text{mes} C^7$, $\sigma_1 > 0$ and $R \leq R_0$, then

$$\inf_{D \cap C^6} v\,(x,t) \geq \eta_4(\gamma, n, \sigma_1). \tag{35}$$

Proof. Without loss of generality, we will assume that $mes_{n-1} \partial D = 0$. Consider a cylinder

$$C^7(\rho_0) = C_{R:1-\rho_0}^{t^0 - b(1-\rho_0)R^2, t^0 - \frac{b}{2}(1+\rho_0)R^2}(x^0), \quad \rho_0 \in \left(0, \frac{1}{4}\right].$$

We choose and fix $\rho_0 > 0$ such that

$$\text{mes}\big(C^7 \setminus C^7(\rho_0)\big) = \frac{\sigma_1}{2} \cdot \text{mes} C^7.$$

It is clearly shown that ρ_0 depends only on n and σ_1. Let us denote by E^0 the set of interior intersection points of $E_R \cap C^7(\rho_0)$. Then

$$\begin{aligned} \text{mes}\, E^0 &\geq \text{mes} E_R - \text{mes}\big(C^7 \setminus C^7(\rho_0)\big) \geq \\ &\geq \sigma_1 \cdot \text{mes} C^7 - \frac{\sigma_1}{2} \cdot \text{mes} C^7 = \frac{\sigma_1}{2} \cdot \text{mes} C^7. \end{aligned} \tag{36}$$

Let $(x`, t')$ be an arbitrary point from E^0, and

$$C_\nu^5(x`, t') = C_{R:8\nu}^{t` - b(\nu R)^2, t`}(x`),\; C_\nu^6(x`, t') = C_{R:\nu}^{t` - \frac{b}{4}(\nu R)^2, t`}(x`),$$

$$C_\nu^7(x`, t') = C_{R:\nu}^{t` - b(\nu R)^2, t` - \frac{b}{2}(\nu R)^2}(x`),$$

where $\nu \in (0, \rho_0]$ is such that $C_\nu^5(x`, t') \in C^7$. We denote by $\nu(x`, t`)$ the exact upper edge of those $\nu \in (0, \rho_0]$ for which the following is true

$$mes\left(D \cap C_\nu^5(x`, t')\right) \leq \delta \cdot mes C_\nu^5(x`, t')\,,$$

where δ is the constant of Lemma 1.4.

There are two possible cases:

1. $\nu(x`, t`) = \rho_0$,
2. $\nu(x`, t`) < \rho_0$.

Let case 1) take place. Then according to Corollary 2.3 either $D \cap C_{\rho_0}^6(x`, t') = \emptyset$, either,

$$\inf_{D \cap C_{\rho_0}^6(x`, t')} v(x,t) \geq \eta_3``.$$

Let $D' = \left\{(x,t) \quad \middle|\, (x,t) \in D, v(x,t) < \eta_3``\right\}$. From the above, it follows that $C^7 \setminus D`$ contains a cylinder $C_{\rho_0}^6(x`, t')$. Applying to the function $\frac{v(x,t)}{\eta_3``}$ Corollary 1.3, we obtain

$$\inf_{D \cap C_{\rho_0}^6(x`, t')} \frac{v(x,t)}{\eta_3``} \geq \eta_2``,$$

and the constant $\eta_2``$ only depends on γ, n and σ_1. Given that $v(x,t) \geq \eta_3`$ for $(x,t) \in D \setminus D'$, we conclude

$$\inf_{D\cap C^6} v(x,t) \geq \eta_2`` \cdot \eta_3``.$$

Thus, if case 1) holds, then Lemma 1.5 is proved.

Let case 2) take place. If for some other point $(x'',t'') \in E^0$ the case 1) is satisfied, then again Lemma 1.5 is proved. So it remains to consider the situation when for any point $(x,t) \in E^0$ there is a case 2). Let's cover E^0 with cylinders $C^6_{\nu(x,t)}(x,t) \in E^0$ and choose from this coverage the countable under covered $\{C^6_{\nu_k}(x^k,t^k)\}, k = 1,2,...,$ of finite multiplicity $q_1(n)$ so that at that

$$\text{mes}\left[\bigcup_{k=1}^{\infty}\left(\text{D}\cap C^5_{\nu_k}(x^k,t^k)\right)\right] \leq q_2(n)\cdot \text{mes}\left[\bigcup_{k=1}^{\infty}\left(\text{D}\cap C^6_{\nu_k}(x^k,t^k)\right)\right].$$

By Corollary 2.3, for every natural k, either $\text{D}\cap C^6_{\nu_k}(x^k,t^k) = \emptyset$, either

$$\inf_{D\cap C^6_{\nu_k}(x^k,t^k)} v(x,t) \geq \eta_3``.$$

Moreover, in consideration of the (36)

$$\text{mes}\left[\bigcup_{k=1}^{\infty}\left(\text{D}\cap C^6_{\nu_k}(x^k,t^k)\right)\right] \geq \frac{1}{q_2}\text{mes}\left[\bigcup_{k=1}^{\infty}\left(\text{D}\cap C^5_{\nu_k}(x^k,t^k)\right)\right] \geq$$

$$\geq \frac{1}{q_1q_2}\cdot\sum_{k=1}^{\infty}\text{mes}\left(\text{D}\cap C^5_{\nu_k}(x^k,t^k)\right) =$$

$$= \frac{\delta}{q_1q_2}\cdot\sum_{k=1}^{\infty}\text{mes}C^5_{\nu_k}(x^k,t^k) \geq \frac{\delta}{q_1q_2}\cdot\sum_{k=1}^{\infty}\text{mes}C^6_{\nu_k}(x^k,t^k) \geq$$

$$\geq \frac{\delta}{q_1q_2}\cdot \text{mes } E^0 \geq \frac{\delta\sigma_1}{2q_1q_2}\cdot\text{mes } C^7. \tag{37}$$

We denote $\frac{v(x,t)}{\eta_3``}$ by $v_1(x,t)$ and let

$$D_1 = \{x,t) : (x,t) \in D, v_1(x,t) < 1\}.$$

Then if $E^1_R = C^7 \backslash D_1$, then according to the condition on E_R and (37)

$$\text{mes } E^1_R \geq \sigma_1\cdot\text{mes } C^7 + \frac{\delta\sigma_1}{2q_1q_2}\cdot\text{mes } C^7 = \sigma_1\cdot\left(1+\frac{\delta}{2q_1q_2}\right)\text{mes } C^7 .$$

Let's apply to the function $v_1(x,t)$ the same procedure as for the function $v(x,t)$. Then we can prove either Lemma 2.5 or we obtain that if

$$D_2 = \left\{x,t) : (x,t) \in D, \frac{v_1(x,t)}{\eta_3``} < 1\right\}, E^2_R = C^7\backslash D_2 ,$$

then

$$\text{mes } E^2_R \geq \sigma_1\cdot\left(1+\frac{\delta}{2q_1q_2}\right)^2\cdot\text{mes } C^7.$$

Let's continue the process following. Clearly, in this chain, alternative 2) cannot repeat more than $l_0(n,\delta,\sigma_1)$ times, where l_0 is the smallest natural number for which

$$\sigma_1 \cdot \left(1 + \frac{\delta}{2q_1 q_2}\right)^{l_0} > 1.$$

Corallary 2.4. Let C^5 be a region D having limit points on $\Gamma(C^5)$ and intersecting C^6. Let D further define a positive L-subelliptic-parabolic function $u(x,t)$, continuous in $\overline{D}$ and going to zero on $\Gamma(D) \cap C^5$. Then if $E_R = C^7 \setminus D$, $\text{mes} E_R \geq \sigma_1 \text{mes}\, C^7, \sigma_1 > 0$, and $R \leq R_0$, then.

$$\sup_D u\,(x,t) \geq (1 + \eta_4) \cdot \sup_{D \cap C^6} u\,(x,t). \qquad (38)$$

Proof. Let $v(x,t) = 1 - \frac{u(x,t)}{M}$, where $M = \sup_D u\,(x,t)$. Applying Lemma 2.5 to the function $v(x,t)$, we obtain that

Hence the required estimate of (38) follows.

Let $C^8 = C_{R:9}^{-bR^2, -\frac{3b}{4}R^2(0)}$.

Lemma 2.6. Let C^3 contain a region D having limit points on the parabolic boundaries of both cylinders C^1 and C^2. Let D further define a positive L - subelliptic-parabolic function $u(x,t)$, continuous in $\overline{D}$ and going to zero on $\Gamma(D) \cap C^3$. Then, if $H_R = C^8 \backslash D$ and $\text{mes} H_R \geq \sigma \cdot C^8, \sigma > 0$, then at $R \leq R_0$,

$$\sup_D u \geq (1 + \eta(\gamma, n, \sigma)) \cdot \sup_{D \cap \Gamma(C^4)} u. \qquad (39)$$

Proof. Without loss of generality, we can consider that $\sup_{D \cap \Gamma(C^4)} u = 1$. Let $(x^*, t^*) \in D \cap \Gamma(C^4)$ be the point at which $u(x^*, t^*) = 1$.

Suppose at the beginning that $(x^*, t^*) \in F(C^4)$, i.e., $(x^*, t^*) = (x^*, t^0)$, where $t^0 = -\frac{bR^2}{8}$. On $F(C^4)$, choose the minimum number of points $(x^1, t^0), \dots, (x^m, t^0)$ such that

1. $\bar{C}^8 \subset \bigcup_{i=1}^m C^7(x^i, t^0)$;
2. One of the points (x^i, t^0) coincides with the point (x^*, t^0);
3. for any $i, 1 \leq i \leq m$, will be found $j, 1 \leq j \leq m$ such that $x^j \in \partial E_{\frac{R}{A^m}}^{x^i}(1)$, where the constant $A(n) > 1$ will be selected later.

Clearly, the number m depends only on n. From the properties of the coating it follows, that for any $i_0, 1 \leq i_0 \leq m$, there is a chain $(x^{i_1}, t^0), \dots, (x^{i_k}, t^0)$ such that $(x^{i_k}, t^0), \dots, (x^*, t^0), x^{i_{e+1}} \in \partial E_{\frac{R}{A^m}}^{x^{i_l}}(1)$, $l = 0, 1, \dots, k-1$.

From the condition on H_R we conclude the existence of $i_0, 1 \leq i_0 \leq m$ such that

$$\text{mes}\left(H_R \cap C^7\left(x^{i_0}, t^0\right)\right) \geq \frac{\text{mes} H_R}{m} \geq \frac{\sigma}{m} \cdot \text{mes}\, C^8. \qquad (40)$$

It's easy to see that $\text{mes}\, C^8 \geq \text{mes}\, C^7\left(x^{i_0}, t^0\right)$. Therefore, from the (40) it follows that

$$\operatorname{mes}\left(H_R \cap C^7\left(x^{i_0}, t^0\right)\right) \geq \frac{\sigma}{m} \cdot \operatorname{mes} C^7\left(x^{i_0}, t^0\right). \qquad (41)$$

Let $\delta_1 = \frac{\eta_4}{2(1+\eta_4)}$, where the constant is η_4 of lemma 2.5 is taken at $\sigma_1 = \frac{\sigma}{m}$. Let's assume that

$$\sup_{D \cap C^6\left(x^{i_0}, t^0\right)} u(x,t) \geq 1 - \delta_1.$$

Then according to Corollary 2.4 and (41)

$$\sup_D u(x,t) \geq (1+\eta_4) \sup_{D \cap C^6\left(x^{i_0}, t^0\right)} u(x,t) \geq (1+\eta_4) \cdot (1-\delta_1) =$$

$$= 1 + \frac{\eta_4}{2} = \left(1 + \frac{\eta_4}{2}\right) \cdot \sup_{D \cap \Gamma(C^4)} u(x,t),$$

and in this case the statement of Lemma 2.6 is proved.

Now, let's say

$$u(x,t) < 1 - \delta_1, \ \ (x,t) \ \in \cap \ C^6\left(x^{i_0}, t^0\right).$$

Consider the function $v_1(x,t) = u\,(x,t) - 1 + \delta_1$. It is not difficult to see that the function $v_1(x,t)$ is L- subelliptic- parabolic in D, since $\delta_1 < 1$.

Let's $D_1 = \{(x,t) \colon (x,t) \in D, v_1(x,t) > 0\}$. The assumption is that the cylinder $C^6\left(x^{i_0}, t^0\right)$ located in addition to D_1.

For $(x`, t') \in \Gamma(C^4)$ denote by $c^i_{R'}(x`, t`)$ cylinder $c^i(x`, t`); i = 5, 6, 7$, highlighting that in it. $R = R'$ . Let's find one now $A > 1$, that $C^5_R(x`, t`) \subset C^6_{AR}(x`, t`)$. Clearly, it is enough for the inclusion to be fair that

$$bR^2 \leq b\frac{(AR)^2}{4}, 8\omega_i^{-1}(R) \leq \omega_i^{-1}(AR), i \ = \ 1, \ldots, n.$$

The last inequalities are satisfied if we fix $A = \max\{2, \alpha\}$.

Now, let's say $\left(x^{i_1}, t^0\right), \ldots, \left(x^{i_k}, t^0\right)$ above-mentioned chain. By construction $C^7_{\frac{R}{A}}\left(x^{i_1}, t^0\right) \backslash D_1$ contains a cylinder $C^{t`-b\left(\frac{R}{A}\cdot\rho_1\right)^2, t`}_{\frac{R}{A}\cdot\rho_1}(x`)$, where ρ_1 depends only on n.

Let's $\sigma_0 = \frac{\eta_2}{2(1+\eta_2)}$, where the constant η_2 of Corollary 2.1 is taken at $\rho = \rho_1$. Let's assume that sup

$$\sup_{D_1 \cap C^6_{\frac{R}{A}}\left(x^{i_1}, t^0\right)} v_1(x,t) \geq \delta_1(1 - \sigma_0),$$

i.e.

$$\sup_{D_1 \cap C^6_{\frac{R}{A}}\left(x^{i_1}, t^0\right)} u(x,t) \geq 1 - \delta_1\sigma_0.$$

Using Corollary 2.1, we obtain

$$\sup_{D_1 \cap C^6_R\left(x^{i_1}, t^0\right)} v_1(x,t) \geq (1+\eta_2) \sup_{D_1 \cap C^6_{\frac{R}{A}}\left(x^{i_1}, t^0\right)} v_1(x,t) \geq$$

$$\geq (1+\eta_2)\cdot\delta_1(1-\sigma_0).$$

Thus

$$\sup_D u(x,t) \geq \sup_{D_1\cap C_R^6(x^{i_1},t^0)} u(x,t) \geq (1-\delta_1) +$$

$$+(1+\eta_2)\cdot\delta_1\cdot(1-\sigma_0) = 1+\frac{\delta_1\eta_2}{2} = \left(1+\frac{\delta_1\eta_2}{2}\right)\sup_{D\cap\Gamma(C^4)} u(x,t),$$

and in this case the statement of Lemma 2.6 is proved.

Let's

$$u(x,t) < 1-\delta_1\sigma_0,\ \ (x,t)\in D_1\cap C_R^6\left(x^{i_1},t^0\right),$$

then consider L-subelliptic-parabolic in D function $v_2(x,t) = u(x,t)-(1-\delta_1\sigma_0)$.

Let's $D_2 = \{(x,t):(x,t)\in D, v_2(x,t)>0\}$. The assumption is that the cylinder $D_1\cap C^6_{\frac{R}{A^2}}\left(x^{i_1},t^0\right)$ is located in addition to D_2. If now

$$\sup_{D_1\cap C^6_{\frac{R}{A^2}}(x^{i_2},t^0)} v_2(x,t) \geq \delta_1\sigma_0(1-\sigma_0),$$

i.e.

$$\sup_{D\cap C^6_{\frac{R}{A^2}}(x^{i_2},t^0)} u(x,t) \geq 1-\delta_1\sigma_0^2,$$

then, applying Corollary 2.1, we obtain

$$\sup_D u(x,t) \geq \sup_{D\cap C^6_{\frac{R}{A^2}}(x^{i_2},t^0)} u(x,t) \geq 1-\delta_1\sigma_0 +$$

$$+(1+\eta_2)\cdot\delta_1\sigma_0\cdot(1-\sigma_0) = 1+\frac{\delta_1\sigma_0\eta_2}{2} = \left(1+\frac{\delta_1\sigma_0\eta_2}{2}\right)\sup_{D\cap\Gamma(C^4)} u(x,t),$$

and in this case the statement of Lemma 2.6 is proved.

If, however $u(x,t) < 1-\delta_1\sigma_0^2$ at $(x,t)\in D\cap C^6_{\frac{R}{A^2}}\left(x^{i_2},t^0\right)$, then we will continue the process as follows. No later than the k-th step, we prove Lemma 2.6, since $u\left(x^{i_k},t^0\right) = u(x^*,t^0) = 1$. So Lemma 2.6 is proved if $(x^*,t^*)\in\bar{F}(C^4)$.

Now, let's say $(x^*,t^*)\in S(C^4)$ and $t^*>t^0$. Clearly $x^*\in\partial E_R^0(9)$. From the above considerations it follows that either Lemma 2.6 is proved, either $u\,(x,t) < \delta_1\sigma_0^m$ for $(x,t)\in D\cap C^6_{\frac{R}{A^m}}(x^*,t^0)$.

Let's choose on the segment I, connecting the points (x^*,t^0) and (x^*,t^*), minimum number of points $(x^*,t^1),\dots,(x^*,t^p)$ so that

4. $I\subset\bigcup_{i=1}^p \bar{C}^6_{\frac{R}{A^m}}\left(x^*,t^i\right);\ t^p = t^*$;
5. at the intersection of $\bar{C}^6_{\frac{R}{A^m}}\left(x^*,t^i\right)\cap\bar{C}^6_{\frac{R}{A^m}}\left(x^*,t^{i+1}\right)$ cylinder is contained $C^7_{\frac{R}{A^m}}\left(x^*,t^{i+1}\right)$, $i=0,1,\dots,p-1$.

Clearly p depends only on n. Consider the function

$$W_1(x,t) = u\,(x,t)-1+\delta_1\cdot\sigma_2^m,$$

where $\sigma_2 = \min\left\{\sigma_0, \frac{\bar{\eta}_2}{2(1+\bar{\eta}_2)}\right\}, \bar{\eta}_2$ -constant η_2 of Corollary 2.1, taken at $\rho = A^{-1}$.

Let's $D^1 = \{(x,t): (x,t) \in D, W_1(x,t) > 0\}$. By assumption, the cylinder $C^6_{\frac{R}{A^m}}(x^*, t^0)$ is located in the supplement of D^1. If

$$\sup_{D^1 \cap C^6_{\frac{R}{A^m}}(x^*,t^1)} W_1(x,t) \geq \delta_1 \sigma_2^m \cdot (1-\sigma_2),$$

i.e.

$$\sup_{D^1 \cap C^6_{\frac{R}{A^m}}(x^*,t^1)} u(x,t) \geq 1 - \delta_1 \sigma_2^{m+1},$$

then, applying Corollary 2.1, we obtain

$$\sup_{D^1 \cap C^6_{\frac{R}{A^{m-1}}}(x^*,t^1)} W_1(x,t) \geq (1+\bar{\eta}_2) \cdot \delta_1 \cdot \sigma_2^m \cdot (1-\sigma_2).$$

Thus

$$\sup_D u(x,t) \geq \sup_{D^1 \cap C^6_{\frac{R}{A^{m-1}}}(x^*,t^1)} u(x,t) \geq$$
$$\geq 1 - \delta_1 \sigma_2^m + (1+\bar{\eta}_2) \cdot \delta_1 \sigma_2^m \cdot (1-\sigma_2) \geq$$
$$\geq 1 + \frac{\delta_1 \sigma_2^m \bar{\eta}_2}{2} = \left(1 + \frac{\delta_1 \sigma_2^m \cdot \bar{\eta}_2}{2}\right) \cdot \sup_{D \cap \Gamma(C^4)} u(x,t),$$

and in this case the statement of Lemma 2.6 is proved.

If $u(x,t) < 1 - \delta_1 \sigma_2^{m+1}$ at $(x,t) \in D^1 \cap C^6_{\frac{R}{A^m}}(x^*, t^1)$, then we continue the process as follows. At the latest p-step, we prove Lemma 2.6, since $u(x^*, t^p) = (x^*, t^*) = 1$.

The estimate of (39) is completely proved.

Theorem 2.1. Let C^1 be a region D that has limit points on $\Gamma(C^1)$ and intersects C^4. Let D then define a positive L-subelliptic-parabolic function $u(x,t)$, which is continuous in $\bar{D}$ going to zero on $\Gamma(D) \cap C^1$. Then, if with respect to the coefficients of the operator L the following conditions are satisfied (2) - (4) , then at $H_R = C^8 \setminus D$, $\mathrm{mes} H_R \geq \sigma \mathrm{mes} C^8$ and $R \leq R_0$

$$\sup_D u(x,t) \geq (1+\eta) \cdot \sup_{D \cap C^4} u(x,t).$$

The statement of Theorem 2.1 follows from Lemma 2.6 and the maximum principle.

3. The oscillation theorem and the internal a priori estimate of the Holder norm of solutions

Theorem 3.1. Let $D \subset R_{n+1}$, $n \geq 1, (0,0) \in D$ in a bounded domain D have coefficients of the operator L which satisfy the conditions (2)-(4). Then, if $u(x,t)$ is the solution of the equation (1) in D and $\bar{C}^1 \subset D$, then at $R \leq R_0$

$$\operatorname*{osc}_{C^1} u(x,t) \geq \left(1+\frac{\eta}{2}\right) \cdot \operatorname*{osc}_{C^4} u(x,t), \tag{42}$$

where η is the constant of Theorem 2.1 at $\sigma = \frac{1}{2}$.

Proof. In the proof suggested below, only Theorem 2.1.1 will be used. In this plan, if

$$M_1 = \sup_{C^1} u\,, m_1 = \inf_{C^1} u\,, M_2 = \sup_{C^4} u\,, m_2 = \inf_{C^4} u$$

and the inequality of the form (42) is valid for the function

$$\vartheta(x,t) = u(x,t) - \frac{M_2+m_2}{2},$$

then it is fulfilled for the function $u(x,t)$. But

$$\sup_{C^4} \vartheta(x,t) = \sup_{C^4} u - \frac{M_2+m_2}{2} = M_2 - \frac{M_2+m_2}{2} = \frac{M_2-m_2}{2},$$
$$\inf_{C^4} \vartheta(x,t) = \inf_{C^4} u - \frac{M_2+m_2}{2} = m_2 - \frac{M_2+m_2}{2} = -\frac{M_2-m_2}{2}.$$

Furthermore, it can always be considered that $M_2 - m_2 > 0$, otherwise inequality (42) is obvious. Therefore, without loss of generality, we will supposed that the $M_2 = 1, m_2 = -1$, i.e. $\operatorname*{osc}_{C^4} u = 2$.

Let`s

$$D^+ = \{(x,t):\ (x,t) \in C^1, u(x,t) > 0\},$$
$$D^- = \{(x,t) : (x,t) \in C^1, u(x,t) < 0\}.$$

Obviously, both of these sets are not empty

At least one of the following inequalities is satisfied

1) $\operatorname{mes}(C^8 \backslash D^+) \geq \frac{1}{2} \operatorname{mes} C^8$,
2) $\operatorname{mes}(C^8 \backslash D^-) \geq \frac{1}{2} \operatorname{mes} C^8$.

Let the case 1) take place for definiteness. We note that alternative 2) reduces to 1) multiplying the solution $u(x,t)$ by (-1). Let's denote by D' that connected component of the set D^+ which contains the point $(x^0,t^0) \in \Gamma(C^4)$, where $u(x^0,t^0) = 1$. By applying Theorem 2.1 to the function $u(x,t)$ in D' with the constant $= \frac{1}{2}$, we obtain that

$$M_1 \geq (1+\eta)M_2 = 1+\eta,$$

i.e.

$$M_1 - m_1 \geq 1 + \eta - m_1 \geq 1 + \eta - m_2 =$$
$$= 2 + \eta = 2\left(1+\frac{\eta}{2}\right) = \left(1+\frac{\eta}{2}\right) \cdot (M_2 - m_2)$$

and the required estimate of (42) is proved.

Corollary 3.1. Let

$$C_R(x^0,t^0) = C_{R:9}^{t^0 - \frac{bR^2}{8}, t^0}(x^0), \quad \nu = \max\left\{3, \frac{\beta}{\alpha}\right\}.$$

Then, if the conditions of Theorem 2.1 are satisfied and $\bar{C}_{\nu R}(0,0) \subset D$, then at $R \leq R_0$

$$\underset{C_{\nu R}(0,0)}{\operatorname{osc}} u(x,t) \geq \left(1+\frac{\eta}{2}\right) \underset{C_R(0,0)}{\operatorname{osc}} u(x,t). \tag{43}$$

For the proof it is enough to notice that at the chosen ν there is an inclusion of $C^1 \subset C_{\nu R}(0,0)$.

Corollary 3.2. An estimate of the form (43) is also valid in cylinders $C_{\nu R}(x^0,t^0)$ and $C_R(x^0,t^0)$ respectively, unless $R \leq R_0$and $C_{\nu R}(x^0,t^0) \subset (D \cup \gamma(D)$. Here $\gamma(D)$ is the top cover of the region D. Indeed, if $(x^0,t^0) \in \bar{C}^4$, then we should repeat almost verbatim the reasoning of paragraph 1 and the proof of Theorem 3.1. If $(x^0,t^0) \in \bar{C}^4$, then we can apply the corresponding result for uniformly parabolic equations of second order (see e.g. [7]).

We denote by $C_\beta(D), 0<\beta<1$ the Banach space of functions $u(x,t)$ defined on D, with finite norm

$$\|u\|_{C_\beta(D)} = \sup_D |u| + \sup_{\substack{(x,t),(y,\tau)\in D\\(x,t)\neq(y,\tau)}} \frac{|u(x,t)-u(y,\tau)|}{\left(|x-y|+\sqrt{|t-\tau|}\right)^\beta} .$$

Let`s $\rho>0, D^\rho = \{(x,t) : (x,t) \in D, C_\rho\ (x,t) \subset D\}$.

Theorem 3.2. Let $u(x,t)$ of the equation (1) be defined in the region D, whose coefficients satisfy the conditions (2)-(4). Then for any $\rho > 0$ there exist constants $\beta(\gamma,n)$, $c_{2.2.1}(\gamma,n,\rho)$ such that

$$\|u\|_{C^\beta(D^\rho)} \leq c_{2.2.1} \cdot \|u\|_{C(D)}. \tag{44}$$

Proof. Let (x^1,t^1) and (x^2,t^2) be two arbitrary points from the D^ρ. For definiteness, we will assume that $t^1<t^2$. Let us fix an arbitrary small enough $\rho > 0$. Two cases are possible:

1. $(x^2,t^2) \in C_\rho(x^1,t^1)$,
2. $(x^2,t^2) \overline{\in} C_\rho(x^1,t^1)$.

Let us first consider case 1). Let for $m=0,1,2,\dots,C(m)$ denote the cylinder $C_{\rho\nu^{-m}}(x^1,t^1)$. Clearly, there exists a non-negative integer m_0 for which

$$(x^2,t^2) \in C(m_0), (x^2,t^2) \overline{\in} C(m_0+1). \tag{45}$$

It follows from (45) that either $x^2 \overline{\in} E^{x^1}_{\rho\nu^{-m_0-1}}(9)$, either $t^1-t^2 \geq \frac{b\rho^2\cdot\nu^{-2m_0-2}}{8}$. If $x^2 \overline{\in} E^{x^1}_{\rho\nu^{-m_0-1}}(9)$, then

$$\sum_{i=1}^n \frac{\left(x_i^1-x_i^2\right)^2}{\left(\omega_i^{-1}(\rho\nu^{-m_0-1})\right)^2} \geq 81.$$

Thus there you'll find i_0, $1 \leq i_0 \leq n$ such that

$$\left|x_{i_0}^1 - x_{i_0}^2\right| \geq \frac{9}{\sqrt{n}} \omega_{i_0}^{-1}(\rho\nu^{-m_0-1}).$$

So, in any case

$$|x^1-x^2| + \sqrt{t^1-t^2} \geq \frac{9}{\sqrt{n}} \omega_i^{-1}(\rho\nu^{-m_0-1}) +$$

$$+\sqrt{\frac{b}{8}}\cdot\rho\nu^{-m_0-1}\geq c_{2.2}(\gamma,n)\omega_i^{-1}(\rho\nu^{-m_0-1}). \qquad (46)$$

Applying now successively Theorem 2.1 to the function $u(x,t)$ in cylinders $C(i)$ and $C(i+1), i=0,1,\dots,m_0-1$, we obtain

$$\operatorname*{osc}_{C(0)} u \geq \left(1+\frac{\eta}{2}\right)^{m_0}\cdot\operatorname*{osc}_{C(m_0)} u\,,$$

i.e.

$$\operatorname*{osc}_{C(m_0)} u \leq \frac{1}{\left(1+\frac{\eta}{2}\right)^{m_0}}\cdot\operatorname*{osc}_{C(0)} u(x,t)\leq$$

$$\leq\frac{\left(1+\frac{\eta}{2}\right)}{\left(1+\frac{\eta}{2}\right)^{m_0+1}}\cdot\operatorname*{osc}_{D} u \leq\frac{2\cdot\left(1+\frac{\eta}{2}\right)}{\left(1+\frac{\eta}{2}\right)^{m_0+1}}\cdot\|u\|_{C(D)}.$$

Thus, taking into account (45), we conclude that

$$|u(x^1,t^1)-u(x^2,t^2)|\leq\frac{c_{2.3}(\gamma,n)}{\left(1+\frac{\eta}{2}\right)^{m_0+1}}\cdot\|u\|_{C(D)}. \qquad (47)$$

Let us denote $=1+\frac{\eta}{2}$, then

$$p^{m_0+1}=\nu^{\log_\nu p^{m_0+1}}=\nu^{(m_0+1)\log_\nu p}=\nu^{(m_0+1)\cdot\beta},$$

where $\beta=\log_\nu p$. Taking into account (46), we obtain

$$p^{-(m_0+1)}=\nu^{-(m_0+1)\beta}=\frac{\left[c_{2.2}\cdot\omega_i^{-1}(\rho\nu^{-m_0-1})\right]^\beta}{\nu^{(m_0+1)\beta}\cdot\left[c_{2.2.2}\cdot\omega_i^{-1}(\rho\nu^{-m_0-1})\right]^\beta}\leq$$

$$\leq\frac{\left[|x^1-x^2|+\sqrt{t^1-t^2}\right]^\beta}{\left[c_{2.2.2}\cdot\nu^{m_0+1}\cdot\omega_i^{-1}(\rho\nu^{-m_0-1})\right]^\beta}.$$

Using the latter estimate in (47), we arrive at the inequality

$$|u(x^1,t^1)-u(x^2,t^2)|\leq$$

$$\leq\frac{c_{2.3}\cdot\left[|x^1-x^2|+\sqrt{t^1-t^2}\right]^\beta\cdot\|u\|_{C(D)}}{\left[c_{2.2.2}\cdot\nu^{m_0+1}\cdot\omega_i^{-1}(\rho\nu^{-m_0-1})\right]^\beta}. \qquad (48)$$

Let alternative 2) now take place. Then either $x^2\,\overline{\in}\,E_\rho^{x^1}(9)$, either $t^1-t^2\geq\frac{b\rho^2}{8}$. Proceeding in the same way as in deriving equation (45), we obtain that in any case

$$|x^1-x^2|+\sqrt{t^1-t^2}\geq c_{2.4}(\gamma,n)\cdot\omega_i^{-1}(\rho)$$

Therefore, taking into account (48), we have

$$|u(x^1,t^1)-u(x^2,t^2)|\leq 2\cdot\|u\|_{C(D)}\leq$$

$$\leq \frac{2\left(|x^1-x^2|+\sqrt{t^1-t^2}\right)^\beta}{\left(c_{2.4}\cdot\omega_i^{-1}(\rho)\right)^\beta}.\|u\|_{C(D)}. \tag{49}$$

From (48) and (49) we conclude

$$\frac{|u(x^1,t^1)-u(x^2,t^2)|}{\left(|x^1-x^2|+\sqrt{t^1-t^2}\right)^\beta}\leq c_{2.5}\cdot\|u\|_{C(D)}, \tag{50}$$

where

$$c_{2.5}=\max\left\{\frac{c_{2.3}}{\left[c_{2.2}\cdot\nu^{m_0+1}\cdot\omega_i^{-1}(\rho\nu^{-m_0-1})\right]^\beta},\frac{2}{\left[c_{2.4}\cdot\omega_i^{-1}(\rho)\right]^\beta}\right\}.$$

Now it is enough to consider the arbitrariness of points (x^1,t^1) and (x^2,t^2) from D^ρ, and from (50) follows the required estimate (43) with $c_{2.1}=c_{2.5}+1$.

Theorem 3.2 is proven.

4. Harnack's inequality for positive solutions

Let $D\subset R_{n+1}$ be a bounded domain, $n\geq 1, (0,0)\in D$, $C^1(x^0,t^0)=C_{R:17}^{t^0-\frac{9bR^2}{8},t^0}(x^0), C^1(0,0)=C^1$, and we're supposed to assume $\bar{C}^1\subset\left(D\cup\gamma(D)\right)$.

Lemma 4.1. Let $(x^0,t^0)\in C^1$ and let $C^1(x^0,t^0)$ contains a region G having limit points on $\Gamma\left(C^1(x^0,t^0)\right)$ and intersecting $C_R(x^0,t^0)$. Let further in G a positive L-subelliptic-parabolic function is defined, continuous in $\bar{G}$ and converging to zero at $\Gamma(G)\cap C^1(x^0,t^0)$. Then, if we consider the operator coefficients L conditions fulfilled (2) – (4), then for anyone $K>0$ there exists $\delta_0(\gamma,n,K)$ such that for $R\leq R_0$ and

$$\text{mes } G\leq\delta_0\cdot\text{mes } C^1(x^0,t^0) \tag{51}$$

there is an evaluation

$$\sup_G u(x,t)\geq K\cdot\sup_{G\cap C_R(x^0,t^0)}u(x,t). \tag{52}$$

Proof. Let η be the constant from Theorem 3.1, taken at $\sigma=\frac{1}{2}$, and let p be the smallest natural number for which

$$(1+\eta)^p\geq K$$

Let's put

$$\delta_0=\left(\frac{36}{17p}\right)^n\cdot\frac{4}{81\cdot 17^n\cdot p}.$$

Let's divide the difference $C^1(x^0,t^0)\backslash C_R(x^0,t^0)$ parabolic boundaries Γ_i of cylinders

$$C_i=C_{R:9+\frac{8i}{p}}^{t^0-\frac{bR^2}{8}\left(1+\frac{8i}{p}\right),t^0}(x^0), i=0,1,\dots,p-1$$

into p parts. It is clear that Γ_0 coincides with $\Gamma\big(C_R(x^0,t^0)\big)$.

Let us denote by $i = 0,1,\ldots,p-1$ $\sup\limits_{G\cap\Gamma_i} u$ through M_i, and let the value of M_i be attained at the point $(x^i,t^i)\in\Gamma_i$.

Let's consider cylinder

$$C_1^{(i)} = C_{R\cdot\frac{4}{p}}^{t^i-\frac{bR^2}{2p},t^i}(x^i), C_2^{(i)} = C_{R\cdot\frac{36}{17p}}^{t^i-\frac{bR^2}{18p},t^i}(x^i),$$

$$C_3^{(i)} = C_{R\cdot\frac{36}{17p}}^{t^i-\frac{4bR^2}{9p},t^i-\frac{bR^2}{3p}}(x^i)\,, i = 0,1,\ldots,p-1.$$

Assuming $C_p = C^1(x^0,t^0)$, we obtain that for $i = 0,1,p-1$

$$C_1^{(i)} \subset C_{i+1}.$$

We have

$$\text{mes}\left(C_3^{(i)}\backslash G\right) \geq \text{mes}\, C_3^{(i)} - \text{mes G}\,, i = 0,1,\ldots,p-1. \quad (53)$$

On the other hand

$$\text{mes}\, C_3^{(i)} = bR^2\cdot\frac{|S_1|}{9p}\cdot\left(\frac{36}{17p}\right)^n\cdot\prod_{i=1}^n \omega_i^{-1}(R), \quad (54)$$

where $|S_1|$ is the volume of a unit n-dimensional sphere.

Furthermore, according to (51)

$$\text{mes G} \leq \delta_0\cdot\text{mes}C^1(x^0,t^0) = \delta_0\cdot\frac{9bR^2}{8}\cdot 17^n\cdot|S_1|\cdot\prod_{i=1}^n\omega_i^{-1}(R). \quad (55)$$

Using (54) - (55) in (33) and considering the choice of δ_0, we conclude

$$\text{mes}\left(C_3^{(i)}\backslash G\right) \geq \frac{1}{2}\text{mes}\, C_3^{(i)}\,, i = 0,1,\ldots,p-1.$$

Hence, according to Theorem 2.1 and the reasoning in Corollary 3.2, it follows that

$$M_{i+1} \geq (1+\eta)M_i, i = 0,1,\ldots,p-1.$$

where $M_p = \sup\limits_G u$. Thus

$$M_p \geq (1+\eta)^p\cdot M_0,$$

and the required estimate (52) is proven.

Lemma 4.1 is proven.

Let G be an arbitrary domain located in $C^1(x^0,t^0)$, where $(x^0,t^0)\in C^1(0,0)$ and $R\leq R_0$.

We denote by $A(G)$ the set of all L-superelliptic-parabolic functions in G, and through $A^+(G)$- the set of all non-negative L-superelliptic - parabolic functions in G.

Let for $\beta\in[0,1]$

$$A_\beta^R(x^0,t^0) = A^+\big(C^1(x^0,t^0)\big)\cap$$
$$\cap\,\{u\colon \text{mes}\,(C^1(x^0,t^0)\cap[(x,t)\colon u(x,t)\geq 1]) \geq \beta\cdot\text{mes}\, C^1(x^0,t^0)\},$$

$$\gamma_\beta^R(x^0,t^0) = \inf\left\{u(x,t^0): x \in E_R^{x^0}\left(\frac{1}{2}\right), u \in A_\beta^R(x^0,t^0)\right\},$$
$$\gamma_\beta^R = \inf \gamma_\beta^R(x^0,t^0), \gamma(\beta) = \varliminf_{R\to 0+} \gamma_\beta^R.$$

Easily see, that $0 \le \gamma(\beta) \le 1$, and function $\gamma(\beta)$ not reduced by β. It can be shown that the function $\gamma(\beta)$ is continuous on [0, 1].

Lemma 4.2. Let $u(x,t) \in A^+\left(C^1(x^0,t^0)\right)$ and $R \le R_0$. If there exist $\beta \in [0,1]$ and $\varepsilon > 0$ such that

$$\text{mes}\{C^1(x^0,t^0) \cap [(x,t) : u(x,t) \ge \varepsilon]\} \ge \beta \cdot \text{mes}\, C^1(x^0,t^0),$$

that

$$u(x,t_0) \ge \varepsilon \cdot \gamma(\beta) \quad \text{for} \quad x \in E_R^{x^0}\left(\frac{1}{2}\right)$$

Proof. The statements of Lemma 4.2 follow from the definition of the function $\gamma(\beta)$.

Lemma 4.3. Let $u(x,t) \in A\left(C^1(x^0,t^0)\right)$, $(x^0,t^0) \in C^1$ and $R \le R_0$. If there are $\beta \in [0,1]$ and $\nu > 0$ such that $u(x^0,t^0) \ge \nu$,

$$\text{mes}\left\{C^1(x^0,t^0) \cap \left[(x,t) : u(x,t) \le \frac{\nu}{2}\right]\right\} \ge \beta \cdot \text{mes}\, C^1(x^0,t^0)$$

that

$$\sup_{C^1(x^0,t^0)} u(x,t) \ge \frac{\nu}{2}\left(1 + \frac{1}{1-\gamma(\beta)}\right). \tag{56}$$

Proof. Suppose that inequality (47) does not hold. Then there exists $\varepsilon_1 > 0$ such that if $W(x,t) = \frac{2u(x,t)}{\nu} - 1$, that

$$\sup_{C^1(x^0,t^0)} W = \frac{1}{1-\gamma(\beta)+\varepsilon_1} = a_1.$$

Let, further $Z(x,t) = 1 - \frac{W(x,t)}{a_1}$, that $Z(x,t) \in A^+\left(C^1(x^0,t^0)\right)$. Moreover, if $u(x,t) \le \frac{\nu}{2}$, that $Z(x,t) \ge 1$. Applying Lemma 4.2 for $\varepsilon = 1$, we obtain $Z(x^0,t^0) \ge \gamma(\beta)$. On the other hand, by condition $W(x^0,t^0) \ge 1$, therefore

$$1 - \frac{1}{a_1} \ge 1 - \frac{W(x^0,t^0)}{a_1} \ge \gamma(\beta),$$

i.e., $a_1 \ge \frac{1}{1-\gamma(\beta)}$, which is impossible, Lemma 4.3 is proven.

Lemma 4.4. The following limit equality holds

$$\lim_{\beta\to 1-0} \gamma(\beta) = 1.$$

Proof. First, let us rephrase the statement of Lemma 4.1. Let $u(x,t) \in A^+(G), u|_{\Gamma(G)\cap C^1(x^0,t^0)} = 1$. Then, for any $K > 0$, there exists $\delta_0(\gamma,n,K)$ such that if , $R \le R_1(\gamma,n,K)$ and condition (51) is satisfied, then

$$\inf_{G\cap C_R(x^0,t^0)} u(x,t) \ge 1 - \frac{1}{K}. \tag{57}$$

Indeed, let $G` = \{(x,t) : u(x,t) < 1\}$,

$$\vartheta(x,t) = 1 - u(x,t) - c_{3.1}\left(t - t^0 + \frac{9bR^2}{8}\right),$$

where the positive constant $c_{3.1}$ will be chosen later. We have

$$\vartheta_i = -u_i, \vartheta_{ij} = -u_{ij}, \vartheta_t = -u_t - c_{3.1}, \vartheta_{tt} = -u_{tt},$$

$$L\vartheta = \sum_{i,j=1}^{n} a_{ij}(x,t)\,\vartheta_{ij} + \varphi(0-t)\vartheta_{tt} - \vartheta_t =$$

$$= -\sum_{i,j=1}^{n} a_{ij}\,u_{ij} - \varphi(0-t)u_{tt} + u_t + c_{2.3.1} = -Lu + c_{2.3.1} \geq c_{2.3.1}. \quad (58)$$

Then the function $\vartheta(x,t)$ will be L-subelliptic-parabolic in G'.

Assume first that $G` \cap C_R(x^0,t^0) \neq \emptyset$. There are two possible cases

1) $\sup\limits_{G`\cap C_R(x^0,t^0)} \vartheta > 0,$

2) $\sup\limits_{G`\cap C_R(x^0,t^0)} \vartheta \leq 0.$

Let case 1) occur. Then, according to Lemma 4.1, if δ_0 corresponds to the constant $2K$, then

$$1 - \inf_{G`} u(x,t) \geq 2K \cdot \left(1 - \inf_{G`\cap C_R(x^0,t^0)} u(x,t) - c_{2.3.1} \cdot \frac{9b}{8} \cdot R^2\right),$$

i.e.

$$\inf_{G`\cap C_R(x^0,t^0)} u(x,t) \geq \frac{2K-1}{2K} - c_{3.1} \cdot \frac{K \cdot 9b}{4} \cdot R^2. \quad (59)$$

We will record now R. Then, from (58) it follows that (57), since $u(x,t) \geq 1$ for $(x,t) \in G\backslash$G`. If $G` \cap C_R(x^0,t^0) \neq \emptyset$, then $u(x,t) \geq 1$ for $(x,t) \in G \cap C_R(x^0,t^0)$. Thus, inequality (57) is proven.

Let us now return to the proof of Lemma 4.4. Suppose that its statement does not hold. Let's fix an arbitrary $\varepsilon > 0$. Then there exists $a \in (0,1)$ such that $\gamma(\beta) < 1 - a$ for $\beta \in (1-\varepsilon, 1)$. Let us set $K = \frac{4}{\alpha}$ in (57) and choose the corresponding δ_0 and R_1. Let's $\varepsilon` = \min\{\varepsilon, \delta_0\}$. By definition of the function $\gamma(\beta)$, there exists $R_2 \leq R_1$ such that $\gamma_\beta^{R_2} < 1 - \frac{\alpha}{2}$ for $\beta \in (1-\varepsilon`, 1)$. Let's fix an arbitrary $\beta_0 \in (1-\varepsilon`, 1)$. Then there will be a point $(x^0,t^0) \in C^1$ (at $R = R_2$), the function $u(x,t) \in A_{\beta_0}^{R_2}(x^0,t^0)$ and the point $x^1 \in E_{R_2}^{x^0}\left(\frac{1}{2}\right)$ such that

$$u(x^1,t^0) < 1 - \frac{a}{4}. \quad (60)$$

Let $D` = \{(x,t) : (x,t) \in C^1(x^0,t^0), u(x,t) < 1\}$. By definition of class $A_{\beta_0}^{R_2}(x^0,t^0)$ we have

$$\text{mes}D` < (1-\beta_0)\text{ mes } C^1(x^0,t^0) \leq \delta_0 \cdot \text{mes } C^1(x^0,t^0).$$

Then, according to (57)

$$\inf_{D`\cap C_{R_2}(x^0,t^0)} u \geq 1-\frac{a}{4}.$$

Considering that $u(x,t)\geq 1$ for $(x,t)\in C_{R_2}(x^0,t^0)\backslash D`$, we conclude that

$$\inf_{C_{R_2}(x^0,t^0)} u \geq 1-\frac{a}{4}$$

and in particular

$$u(x^1,t^0)\geq 1-\frac{a}{4}.$$

The last inequality contradicts (50).

Lemma 4.4 is proven.

Lemma 4.5. Let $R\leq R_0, \sigma\in(0,1], H_j\in\left[\frac{1}{4},1\right], j=1,2;$

$$-2H_1bR^2\leq\tau\leq -H_1bR^2, 4H_2\omega_i^{-1}(R)\leq x_i^2-x_i^1\leq 8H_2\cdot\omega_i^{-1}(R),$$
$$x_i^1+H_2\omega_i^{-1}(R)\leq x_i^0\leq x_i^2-H_2\omega_i^{-1}(R); i=1,\dots,n; x^0\in E_R^0(4),$$
$$N=\left\{(x,t): x_i^1<x_i<x_i^2; i=1,\dots,n, \tau-2H_1bR^2<t<\tau\right\}.$$

Then, if $u(x,t)\in A^+(N)$ and $u(x,\tau-2H_1bR^2)\geq 1$ for $x\in\bar{E}_R^{x^0}(\sigma)$ then there exists $m(\gamma,n,H_1,H_2)$ such that $u(x,\tau)\geq\sigma^m$ for

$$x_i^1+H_0\omega_i^{-1}(R)\leq x_i\leq x_i^2-H_0\omega_i^{-1}(R);\ \ i=1,\dots,n$$

where $H_0=\min\{H_1,H_2\}$.

Proof. First, note that the statements of all four previous lemmas in this paragraph remain valid if, instead of the original cylinder $C^1(x^0,t^0)$, we take the cylinder $C^0(x^0,t^0)=C_{R:17}^{t^0-4bR^2,t^0}(x^0)$. Without loss of generality, we will assume that $x^0=0$ and $2\sigma^2<H_0^2$. We will fix the point $(x^*,\tau)\in\bar{N}$ such that

$$x_i^1+H_0\omega_i^{-1}(R)\leq x_i^*\leq x_i^2-H_0\omega_i^{-1}(R);\ \ i=1,\dots,n.$$

Let us denote $=\frac{H_0^2}{4H_1}, y=\frac{x^*}{2H_1bR^2}$. Consider the set

$$S=\left\{(x,t):\sum_{i=1}^n\frac{[x_i-(t-\tau+2H_1bR^2)y_i]^2}{\left(\omega_i^{-1}(R)\right)^2}<\right.$$
$$\left.<\xi(t-\tau+2H_1bR^2)+\sigma^2;\ \tau-2H_1bR^2<t<\tau\right\}.$$

It is easy to see that the set S is entirely located in the inclined cylinder.

$$S_1=\left\{(x,t):\sum_{i=1}^n\frac{[x_i-(t-\tau+2H_1bR^2)y_i]^2}{\left(\omega_i^{-1}(R)\right)^2}<\right.$$
$$\left.<2\xi H_1+\sigma^2;\ \tau-2H_1bR^2<t<\tau\right\}.$$

On the other hand, at the lower base S_1, i.e., $t=\tau-2H_1bR^2$

$$\sum_{i=1}^n\frac{x_i^2}{\left(\omega_i^{-1}(R)\right)^2}<\frac{H_0^2}{4H_1}\cdot 2H_1+\sigma^2=\frac{H_0^2}{2}+\frac{H_0^2}{2}=H_0^2.$$

Thus, for points $(x,\tau-2H_1bR^2)$ of the lower base S_1, the following inequalities hold

$$|x_i| < H_0\omega_i^{-1}(R), i = 1, \dots, n$$

Given that $x_i^1 \leq -H_2\omega_i^{-1}(R), x_i^2 \geq H_2\omega_i^{-1}(R)$ we conclude that $x_i^1 < x_i < x_i^2, i = 1, \dots, n$. Thus we have shown that S_1 is located on $\overline{N}$.

Next, for the points of the upper base S_1, i.e., when $t = \tau$

$$\sum_{i=1}^{n} \frac{(x_i - 2H_1bR^2y_i)}{\left(\omega_i^{-1}(R)\right)^2} = \sum_{i=1}^{n} \frac{(x_i - x_i^*)^2}{\left(\omega_i^{-1}(R)\right)^2} < H_0^2.$$

Thus, for the specified points, $|x_i - x_i^*| < H_0 \cdot \omega_i^{-1}(R)$, i.e., $x_i < x_i^* + H_0 \cdot \omega_i^{-1}(R) \leq x_i^2$ and $x_i > x_i^* - H_0\omega_i^{-1}(R) \leq x_i^1, i = 1, \dots, n$. Thus, the upper base S_1 is also located on $\overline{N}$. From the convexity of N, it follows that the inclined cylinder S_1 together with the sets S are located on $\overline{N}$. We note that the parabolic boundary S is the union of the sets $\Gamma_1(S)$ and $\Gamma_2(S)$, where

$$\Gamma_1(S) = \left\{(x,t): \sum_{i=1}^{n} \frac{[x_i - (t - \tau + 2H_1bR^2)y_i]^2}{\left(\omega_i^{-1}(R)\right)^2} = \right.$$

$$\left. = \xi(t - \tau + 2H_1bR^2) + \sigma^2;\ \tau - 2H_1bR^2 \leq t \leq \tau\right\}$$

$$\Gamma_2(S) = \left\{(x,t): \sum_{i=1}^{n} \frac{x_i^2}{\left(\omega_i^{-1}(R)\right)^2} = \sigma^2; t = \tau - 2H_1bR^2\right\}.$$

Let us introduce for $(x,t) \in \bar{S}$ the functions

$$Z_i(x,t) = \frac{x_i - (t - \tau + 2H_1bR^2)y_i}{\sqrt{\xi(t - \tau + 2H_1bR^2) + \sigma^2}}, i = 1, \dots, n;$$

$$r(x,t) = \sum_{i=1}^{n} \frac{[x_i - (t - \tau + 2H_1bR^2)y_i]^2}{\left(\omega_i^{-1}(R)\right)^2} / [\xi(t - \tau + 2H_1bR^2) + \sigma^2];$$

$$\Phi(x,t) = \frac{[1 - r(x,y)]^2}{[\xi(t - \tau + 2H_1bR^2) + \sigma^2]^d},$$

where the positive constant d will be chosen later. It is easy to see that $0 \leq r(x,t) \leq 1$ for $(x,t) \in \bar{S}$, where $r|_{\Gamma_1(S)} = 1$.

We have

$$\frac{\partial \Phi}{\partial x_i} = -\frac{1}{[\xi(t - \tau + 2H_1bR^2) + \sigma^2]^d} \cdot 2\big(1 - r(x,t)\big) \cdot \frac{\partial \mathrm{r}}{\partial x_i};$$

$$\frac{\partial^2 \Phi}{\partial x_i^2} = \frac{1}{[\xi(t - \tau + 2H_1bR^2) + \sigma^2]^{d+1}} \times$$

$$\times \left\{2 \cdot \frac{\partial \mathrm{r}}{\partial x_i} \cdot \frac{2[x_i - (t - \tau + 2H_1bR^2)y_i]}{\left(\omega_i^{-1}(R)\right)^2} - 2(1 - r) \cdot \frac{2}{\left(\omega_i^{-1}(R)\right)^2}\right\},$$

$$\frac{\partial \mathrm{r}}{\partial x_i}=\frac{1}{\xi(t-\tau+2H_1bR^2)+\sigma^2}\cdot\frac{2[x_i-(t-\tau+2H_1bR^2)y_i]}{\left(\omega_i^{-1}(R)\right)^2};$$

$$\frac{\partial^2\Phi}{\partial x_i\partial x_j}=-\frac{8\xi_i\xi_j}{[\xi(t-\tau+2H_1bR^2)+\sigma^2]^{d+1}}\cdot\frac{1}{\left(\omega_i^{-1}(R)\right)^2\left(\omega_j^{-1}(R)\right)^2}.$$

$$\frac{\partial \mathrm{r}}{\partial t}=-\frac{2}{\xi(t-\tau+2H_1bR^2)+\sigma^2}\cdot\sum_{i=1}^{n}\frac{y_i[x_i-(t-\tau+2H_1bR^2)y_i]}{\left(\omega_i^{-1}(R)\right)^2}-$$

$$-\frac{\xi}{[\xi(t-\tau+2H_1bR^2)+\sigma^2]^2}\cdot\sum_{i=1}^{n}\frac{[x_i-(t-\tau+2H_1bR^2)y_i]^2}{\left(\omega_i^{-1}(R)\right)^2};$$

$$\frac{\partial^2 r}{\partial t^2}=\frac{2}{\xi(t-\tau+2H_1bR^2)+\sigma^2}\cdot\sum_{i=1}^{n}\frac{y_i^2}{\left(\omega_i^{-1}(R)\right)^2}+$$

$$+\frac{4\xi}{[\xi(t-\tau+2H_1bR^2)+\sigma^2]^2}\cdot\sum_{i=1}^{n}\frac{y_i[x_i-(t-\tau+2H_1bR^2)y_i]}{\left(\omega_i^{-1}(R)\right)^2}+$$

$$+\frac{2\xi^2}{[\xi(t-\tau+2H_1bR^2)+\sigma^2]^3}\cdot\sum_{i=1}^{n}\frac{[x_i-(t-\tau+2H_1bR^2)y_i]^2}{\left(\omega_i^{-1}(R)\right)^2};$$

$$\frac{\partial\Phi}{\partial t}=-\frac{2(1-r)\cdot r_t}{[\xi(t-\tau+2H_1bR^2)+\sigma^2]^d}-\frac{d(1-r)^2\cdot\xi}{[\xi(t-\tau+2H_1bR^2)+\sigma^2]^{d+1}};$$

$$\frac{\partial^2\Phi}{\partial t^2}=-\frac{2[r_t^2-(1-r)\cdot r_{tt}``]}{[\xi(t-\tau+2H_1bR^2)+\sigma^2]^d}+\frac{4\xi d\cdot(1-r)\cdot r_t}{[\xi(t-\tau+2H_1bR^2)+\sigma^2]^{d+1}}+$$

$$+\frac{\xi d^2\cdot(1+d)\cdot(1-r)^2}{[\xi(t-\tau+2H_1bR^2)+\sigma^2]^{d+2}};$$

$$L\Phi(x,t)=\sum_{i,j=1}^{n}a_{ij}(x,t)\frac{\partial^2\Phi}{\partial x_i\partial x_j}+\varphi(0-t)\frac{\partial^2\Phi}{\partial t^2}-\frac{\partial\Phi}{\partial t}\geq$$

$$\geq[\xi(t-\tau+2H_1bR^2)+\sigma^2]^{-d-1}\times$$

$$\times\left\{8\cdot\sum_{i,j=1}^{n}a_{ij}(x,t)\frac{z_iz_j}{\left(\omega_i^{-1}(R)\right)^2\cdot\left(\omega_j^{-1}(R)\right)^2}+(1-r)^2\xi d+\right.$$

$$+2(r-1)\cdot\left[2\cdot\sum_{i=1}^{n}\frac{a_{ij}(x,t)}{\left(\omega_i^{-1}(R)\right)^2}+\xi r-\right.$$

$$-2\sum_{i=1}^{n}\frac{y_i(x_i-(t-\tau+2H_1bR^2)y_i)}{\left(\omega_i^{-1}(R)\right)^2}\Bigg]\Bigg\}, \qquad (61)$$

where $\varphi(0-t)\frac{\partial^2\Phi}{\partial t^2}\geq 0$ (follows from condition (5)).

From condition (20) we obtain

$$\sum_{i,j=1}^{n}a_{ij}(x,t)\frac{z_iz_j}{\left(\omega_i^{-1}(R)\right)^2\cdot\left(\omega_j^{-1}(R)\right)^2}=\left(\sum_{i,j=1}^{n}\frac{a_{ij}^2(x,t)}{\lambda_i(x,t)\lambda_j(x,t)}\right)^{1/2}\times$$

$$\times\left(\sum_{i,j=1}^{n}\lambda_i(x,t)\lambda_j(x,t)\cdot\frac{z_i^2z_j^2}{\left(\omega_i^{-1}(R)\right)^4\cdot\left(\omega_j^{-1}(R)\right)^4}\right)^{1/2}=$$

$$=\sum_{i=1}^{n}\frac{a_{ii}(x,t)}{\lambda_i(x,t)}\cdot\sum_{i=1}^{n}\lambda_i(x,t)\cdot\frac{z_i^2}{\left(\omega_i^{-1}(R)\right)^4}\geq$$

$$\geq\gamma\cdot\sum_{i=1}^{n}\lambda_i(x,t)\cdot\frac{z_i^2}{\left(\omega_i^{-1}(R)\right)^4},$$

$$\sum_{i=1}^{n}\frac{a_{ii}(x,t)}{\left(\omega_i^{-1}(R)\right)^2}\leq\left(\sum_{i=1}^{n}\frac{a_{ii}^2(x,t)}{\lambda_i^2(x,t)}\right)^{1/2}\times$$

$$\times\left(\sum_{i=1}^{n}\lambda_i^2(x,t)\cdot\frac{1}{\left(\omega_i^{-1}(R)\right)^4}\right)^{1/2}=\sum_{i=1}^{n}\frac{a_{ii}(x,t)}{\lambda_i(x,t)}\times$$

$$\times\sum_{i=1}^{n}\frac{\lambda_i(x,t)}{\left(\omega_i^{-1}(R)\right)^2}\leq\frac{1}{\gamma}\cdot\sum_{i=1}^{n}\frac{\lambda_i(x,t)}{\left(\omega_i^{-1}(R)\right)^2}. \qquad (62)$$

On the other hand, for $(x,t)\in S$

$$c_{3.2}(\gamma,n,H_1,H_2)\cdot\left(\frac{\omega_i^{-1}(R)}{R}\right)^2\leq\lambda_i(x,t)\leq$$

$$\leq c_{3.3}(\gamma,n,H_1,H_2)\cdot\left(\frac{\omega_i^{-1}(R)}{R}\right)^2, i=1,\dots,n. \qquad (63)$$

Using (63) and (62), we conclude that

$$\sum_{i,j=1}^{n}a_{ij}(x,t)\frac{z_iz_j}{\left(\omega_i^{-1}(R)\right)^2\cdot\left(\omega_j^{-1}(R)\right)^2}\geq c_{3.4}(\gamma,n,H_1,H_2)r,$$

$$\sum_{i=1}^{n}\frac{a_{ii}(x,t)}{\left(\omega_i^{-1}(R)\right)^2}\le c_{3.5}(\gamma,n,H_1,H_2). \qquad (64)$$

We have, furthermore

$$\left|\sum_{i=1}^{n}\frac{y_i(x_i-(t-\tau+2H_1bR^2)y_i)}{\left(\omega_i^{-1}(R)\right)^2}\right|\le$$

$$\le\left(\sum_{i=1}^{n}\frac{[x_i-(t-\tau+2H_1bR^2)y_i]^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2}\cdot\left(\sum_{i=1}^{n}\frac{y_i^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2}\le$$

$$\le\left[(\xi(t-\tau+2H_1bR^2)+\sigma^2)\cdot\frac{1}{4H_1^2b^2R^4}\cdot\sum_{i=1}^{n}\frac{(x_i^*)^2}{\left(\omega_i^{-1}(R)\right)^2}\right]^{\frac{1}{2}}\le$$

$$\le c_{3.6}(\gamma,n,b,H_1,H_2). \qquad (65)$$

Given that $\xi r\le 1$ and using (64)-(65) in (61), we obtain

$$L\Phi\ge[\xi(t-\tau+2H_1bR^2)+\sigma^2]^{-d-1}\times$$

$$\times\{8c_{3.4}\cdot r+(1-r)^2\xi d-2(1-r)(2\cdot c_{3.5}+1+2c_{3.6})\}. \qquad (66)$$

It follows that there exists $r_0(\gamma,n,H_1,H_2)<1$, sufficiently close to unity, such that

$$8c_{3.4}\ge 2(1-r)(2c_{3.5}+1+2c_{3.6}),$$

if only $r_0\le r\le 1$. If $0\le r<r_0$, then there exists a sufficiently large $d(\gamma,n,H_1,H_2)$ such that

$$(1-r)^2\xi d\ge 2(1-r)(2c_{3.5}+1+c_{3.6}).$$

Let us denote this by d. Then the function $\Phi(x,t)$ is L-subelliptic-parabolic in S.

Let $u(x,t)\in A^+(N)$. Consider the auxiliary function $w(x,t)=u(x,t)-\sigma^{2d}\cdot\Phi(x,t)$. It is clear that $w(x,t)\in A(S)$. Furthermore, $w\ \big|_{\Gamma_1(S)}\ge 0$, since $\Phi\ \big|_{\Gamma_1(S)}\ge 0$. On the other hand

$$w\ \big|_{\Gamma_2(S)}\ge 1-\sigma^{2d}\cdot\Phi\ \big|_{\Gamma_2(S)}=1-(1-r)^2\ge 0.$$

By the maximum principle $w(x,t)\ge 0$ for $(x,t)\in\bar{S}$. In particular, at the point (x^*,τ), where $r=0$, we obtain

$$u(x^*,\tau)\ge\frac{\sigma^{2d}}{(\xi 2H_1bR^2+\sigma^2)^d}\ge\frac{\sigma^{2d}}{(bH_0^2)^d}\ge\sigma^{2d}. \qquad (67)$$

Now it suffices to choose $m=2d$ and the lemma is proven. Let the cylinder be denoted by $C^0(0,0), C^0: C^0=C_{R:17}^{-4bR^2,0}(0)$.

Theorem 4.1. Let $u(x,t)$ be a positive solution of equation (1) in the domain D, and let the conditions (2) - (4) be satisfied with respect to the coefficients of the operator L. Then, if $\bar{C}^0 \subset D$ and $R \le R_0$, then

$$u(0,-bR^2) \le c_{2.3.7}(\gamma,n) \cdot \inf_{x \in E_R^0\left(\frac{1}{4}\right)} u\left(x, -\frac{bR^2}{2}\right). \qquad (68)$$

$Proof$. Let the number m from Lemma 4.5 correspond to $H_1 = \frac{1}{4}$, $H_2 = 1$. Fix this m and find, according to Lemma 4.4, such $\beta \in (0,1)$ that

$$\frac{1}{2}\left(1 + \frac{1}{1-\gamma(1-\beta)}\right) > 2^m. \qquad (69)$$

Let`s asuume that $r \in (0,1), \nu(r) = u(0,-bR^2) \cdot (1-r)^{-m}$,

$$Q(r) = \{\,(x,t): x \in \bar{E}_R^0(r), -bR^2(1+r^2) \le t \le -br^2\},$$

$$g(r) = \max_{Q(r)} u(x,t).$$

Let`s asuume that $(x^*,t^*) \in Q(r_1), g(r_1) = \nu(r_1) = u(x^*,t^*)$,

$$F = \left\{(x,t):\ x \in E_R^{x^*}\left(\frac{1-r_1}{2}\right), t^* - \frac{b(1-r_1^2)}{4}R^2 < t < t^*\right\}.$$

For $(x,t) \in \bar{F}$, we have

$$\left(\sum_{i=1}^{n} \frac{x_i^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{\frac{1}{2}} \le \left(\sum_{i=1}^{n} \frac{(x_i - x_i^*)^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{\frac{1}{2}} +$$

$$+\left(\sum_{i=1}^{n} \frac{(x_i^*)^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{\frac{1}{2}} \le \frac{1-r_1}{2} + r_1 = \frac{1+r_1}{2}.$$

On the other hand

$$1 + r_1^2 + \frac{1-r_1^2}{4} < 1 + \frac{(1+r_1)^2}{4}.$$

Therefore, $\bar{F} \subset Q\left(\frac{1+r_1}{2}\right)$, i.e., for $(x,t) \in \bar{F}$ by virtue of (69) and the fact that $\frac{1+r_1}{2} > r_1$, we have

$$u(x,t) \le g\left(\frac{1+r_1}{2}\right) < \nu\left(\frac{1+r_1}{2}\right) = u(0,-bR^2) \cdot \left(1 - \frac{1+r_1}{2}\right)^{-m} \le$$

$$\le 2^m \cdot \nu(r_1) < \frac{\nu(r_1)}{2} \cdot \left(1 + \frac{1}{1-\gamma(1-\beta)}\right). \qquad (70)$$

If we now assume that

$$\mathrm{mes}\left\{F \cap \left[(x,t):\ u(x,t) \le \frac{\nu(r_1)}{2}\right]\right\} \ge (1-\beta)\mathrm{mes}F,$$

From the equality $u(x^*,t^*) = \nu(r_1)$ and Lemma 4.3, it follows that

$$\sup_F u(x,t) \ge \frac{\nu(r_1)}{2} \cdot \left(1 + \frac{1}{1-\gamma(1-\beta)}\right).$$

The last inequality is impossible due to (70). We used the fact that $u(x,t)$ is a solution to equation (1), i.e., $u \in A(F)$. Thus

$$\operatorname{mes}\left\{F \cap \left[(x,t):\ u(x,t) \le \frac{\nu(r_1)}{2}\right]\right\} < (1-\beta)\operatorname{mes}F,$$

i.e.

$$\operatorname{mes}\left\{F \cap \left[(x,t):\ u(x,t) \le \frac{\nu(r_1)}{2}\right]\right\} \ge \beta\operatorname{mes}F. \qquad (71)$$

Now we use Lemma 4.5. There are two possible cases: $r_1 > \frac{1}{3}$ and $r_1 \in \left(0,\frac{1}{3}\right]$. Let us assume the first case. Let us assume

$$x^0 = \frac{9r_1-1}{8r_1}\,x^*, \sigma = -\frac{bR^2}{2}, -\frac{bR^2}{2} - 2H_1bR^2 = t^*, H_2 = \frac{9r_1-1}{8}$$

It is not difficult to see that $\frac{1}{4} \le H_j \le 1, j = 1,2.$

If now $\sigma = \frac{1-r_1}{8}$, then $E_R^{x^0}(\sigma) \subset E_R^{x^0}\left(\frac{1-r_1}{4}\right)$.

Indeed, let $x \in E_R^{x^0}(\sigma)$, then

$$\sum_{i=1}^{n} \frac{\left(x_i - x_i^0\right)^2}{\left(\omega_i^{-1}(R)\right)^2} < \frac{(1-r_1)^2}{64}.$$

Therefore

$$\left(\sum_{i=1}^{n} \frac{(x_i - x_i^*)^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{\frac{1}{2}} \le \left(\sum_{i=1}^{n} \frac{\left(x_i - x_i^0\right)^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{\frac{1}{2}} +$$

$$+\left(\sum_{i=1}^{n} \frac{\left(x_i^0 - x_i^*\right)^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{\frac{1}{2}} \le \frac{1-r_1}{8} + \frac{1-r_1}{8r_1}\cdot\left(\sum_{i=1}^{n} \frac{(x_i^*)^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{\frac{1}{2}} \le$$

$$\le \frac{1-r_1}{8} + \frac{1-r_1}{8r_1}\cdot r_1 = \frac{1-r_1}{4}.$$

Let $x_i^1 = -2H_2\omega_i^{-1}(R), x_i^2 = 2H_2\omega_i^{-1}(R), i = 1,\dots,n.$ Then, from Lemmas 4.2 and 4.5, taking into account (51), it follows that for $x \in E_R^0\left(\frac{1}{4}\right)$

$$u\left(x, -\frac{bR^2}{2}\right) \ge \left(\frac{1-r_1}{8}\right)^m \cdot \frac{\nu(r_1)}{2}\cdot\gamma(\beta) =$$

$$= \left(\frac{1-r_1}{8}\right)^m \cdot \frac{1}{2}\cdot u(0,-bR^2)\cdot(1-r_1)^{-m}\cdot\gamma(\beta) =$$

$$= 2^{-3m-1}\cdot\gamma(\beta)\cdot u(0,-bR^2). \qquad (72)$$

Let $r_1 \in \left(0,\frac{1}{3}\right]$, and τ, σ and H_1 have the same meaning as above. Let

$$x^0 = \frac{7r_1+1}{8r_1}x^*, H_2 = 1, x_i^1 = -\left(\frac{8r_1}{7r_1+1}+1\right)\omega_i^{-1}(R),$$
$$x_i^2 = \left(\frac{8r_1}{7r_1+1}+1\right)\omega_i^{-1}(R), i = 1,\dots,n.$$

Then, taking into account Lemmas 4.2 and 4.5 and inequality (51), we again arrive at estimate (52). Thus, the required inequality (58) with $c_{3.7} = 2^{3m+1} \cdot \gamma(R)$ is proven.

Corollary 4.1. If the conditions of the theorem are satisfied, then for $R \le R_0$, the following estimate holds

$$u(0,-bR^2) \le c_{3.8}(\gamma,n,b)\cdot \inf_{x\in E_R^0\left(\frac{1}{4}\right)} u\left(x,-\frac{bR^2}{4}\right).$$

Lemma 4.6. Let the conditions of Theorem 4.1 be satisfied. Then, if $\bar{x} \in \partial E_R^0\left(\frac{1}{4}\right)$, then for $R \le R_0$ the following estimate holds

$$u(\bar{x},-bR^2) \le c_{3.9}(\gamma,n.b)\cdot \inf_{\theta\in(0,1)} u\left(\bar{x},-(1-\theta)\frac{bR^2}{4}\right).$$

Proof. Let us fix an arbitrary point $\bar{x} \in \partial E_R^0\left(\frac{1}{4}\right)$. It is easy to see that if $\bar{x} \in \partial E_R^0\left(\frac{1}{4}\right), \bar{x} \in \partial E_R^{\bar{x}}\left(\frac{1}{8}\right)$, then $x \,\overline{\in}\, E_R^0\left(\frac{1}{8}\right)$. Consider the cylinder $C^9 = C_{R;\frac{1}{8}}^{-2bR^2,0}(\bar{x})$. Let us make the following variable substitutions: $y_i = \frac{x_i}{\omega_i^{-1}(R)}, i = 1,\dots,n,\ \tau = R^{-2}t$. Then, under our transformation, C^9 will be a straight circular cylinder $\tilde{C}^9 = B_{\frac{1}{8}}(\bar{y}) \times (-2b,0)$, where $\bar{y}$ is the image of the point $\bar{x}$. It is clear that $\bar{y} \in \partial B_{\frac{1}{4}}(0)$.

Let $\tilde{u}_R(y,\tau)$ be the image of the function $u(x,t)$. Then equation (1) in variables (y,τ) take on the appearance of

$$\tilde{L}_R\tilde{u}_R = \sum_{i,j=1}^{n} a_{ij}^R(y,\tau)\frac{\partial^2\tilde{u}_R}{\partial y_i\partial y_j} + \varphi(0-R^2\tau)\cdot R^2\cdot\frac{\partial^2 u_R}{\partial\tau^2} - \frac{\partial u_R}{\partial\tau} = 0$$

where

$$a_{ij}^R(y,\tau) = \frac{R^2}{\omega_i^{-1}(R)\cdot\omega_j^{-1}(R)}\cdot a_{ij}(y_1\omega_1^{-1}(R),\dots,y_n\omega_n^{-1}(R),\tau R^2).$$

For $(y,\tau) \in \tilde{C}^9$ (i.e. $(x,t) \in C^9$)

$$\gamma \le \sum_{i=1}^{n}\frac{a_{ii}(x,t)}{\lambda_i(x,t)} = \sum_{i=1}^{n}\left(\frac{\omega_i^{-1}(R)}{R}\right)^2\cdot\frac{a_{ii}^R}{\lambda_i(x,t)}. \qquad (73)$$

However, for $x \,\overline{\in}\, E_R^0\left(\frac{1}{8}\right), x \in E_R^{\bar{x}}\left(\frac{1}{8}\right)$, and $|t| \le 2bR^2$ the following inequality holds

$$c_{3.10}(\gamma,n)\left(\frac{\omega_i^{-1}(R)}{R}\right)^2 \le \lambda_i\ (x,t) \le$$

$$\leq c_{3.11}(\gamma,n)\left(\frac{\omega_i^{-1}(R)}{R}\right)^2, i=1,\dots,n. \tag{74}$$

Using (74) in (73), we conclude that

$$\gamma \leq \sum_{i=1}^{n} a_{ii}^R(y,\tau), \gamma_1 \in (0,1]$$

and

$$\sigma = \sup_{\tilde{C}^9}\left[\sum_{i,j=1}^{n}\left(a_{ij}^R(t,\tau)\right)^2 / \left(\sum_{i=1}^{n} a_{ii}^R(y,\tau)\right)^2\right] - \frac{1}{n-e^2} < 0.$$

Then, according to the theorem of I. V. Krykov and M. V. Safonov [11], for uniformly subelliptic-parabolic equations of the second order, we obtain

$$\tilde{u}^R\left(\bar{y},-\frac{b}{2}\right) \leq c_{2.3.12}(\gamma,n)\cdot \inf_{\theta\in(0,1)} \tilde{u}^R\left(\bar{y},-(1-\theta)\frac{b}{4}\right).$$

Now it is enough to return to the variables (x,t).

Lemma 4.6 is proven.

Let $C^{10} = C_{R:\frac{1}{4}}^{-\frac{bR^2}{4},0}(0)$.

Corollary 4.2. Let the conditions of Theorem 4.1 be satisfied. Then, for $R \leq R_0$, the estimate

$$u(0,-bR^2) \leq c_{3.13}(\gamma,n)\cdot \inf_{C^{10}} u(x,t).$$

Indeed, let $(\bar{x},\bar{t})$ be a point on the lateral surface $S(C^{10})$ such that $u(\bar{x},\bar{t}) = \inf_{S(C^{10})} u$. According to Theorem 4.1, we have

$$u(0,-bR^2) \leq c_{3.7}\cdot \inf_{x\in E_R^0\left(\frac{1}{4}\right)} u\left(x,-\frac{bR^2}{2}\right) \leq c_{3.7}\cdot u\left(\bar{x},-\frac{bR^2}{2}\right).$$

Applying Lemma 4.6, we obtain

$$u(0,-bR^2) \leq c_{3.7}\cdot c_{3.9}\cdot \inf_{S(C^{10})} u. \tag{75}$$

On the other hand, according to consequence 4.1

$$u(0,-bR^2) \leq c_{3.8}\cdot \inf_{F(C^{10})} u, \tag{76}$$

where $F(C^{10})$ is the lower base of the cylinder C^{10}.

From (75) and (76) it follows that

$$u(0,-bR^2) \leq c_{3.14}\cdot \inf_{\Gamma(C^{10})} u,$$

where $c_{3.14} = \max\{c_{3.7}, c_{3.9}, c_{3.8}\}$. Now it is enough to apply the maximum principle and the corollary is proven.

Let $C^{11} = C_{R:\frac{1}{4}}^{-2bR^2,-\frac{7bR^2}{4}}(0)$.

Theorem 4.2. Let $u(x,t)$ be a positive solution of equation (1) in the domain D, where the conditions (2) – (4) are satisfied with respect to the coefficients of the operator L. Then, if $\bar{C}^0 \subset D$ and $\leq R_0$, then

$$\sup_{C^{11}} u \leq c_{3.15}(\gamma, n) \cdot \inf_{C^{10}} u. \quad (77)$$

Proof. Consider the cylinders $C^{12} = C_{R:1}^{-3bR^2, -bR^2}(0)$ and $C^{13} = C_{R:\frac{1}{4}}^{-\frac{5bR^2}{4}, -bR^2}(0)$. Let us perform the same coordinate transformation as in the proof of Lemma 4.6. Then the images of C^{11}, C^{12} and C^{13} will be straight circular cylinders $\tilde{C}^{11} = B_{\frac{1}{4}}(0) \times \left(-2b, -\frac{7b}{4}\right), \tilde{C}^{12} = B_1(0) \times (-3b, -b)$ and $\tilde{C}^{13} = B_{1/4}(0) \times \left(-\frac{5b}{4}, -b\right)$, respectively. Proceeding as in the proof of Lemma 2.3.6, we show that the image $\tilde{u}_R(y,\tau)$ of the function $u(x,t)$ satisfies a uniformly elliptic-parabolic equation of the form (1) in $\tilde{C}^{12}$. According to the Harnack inequality for uniformly elliptic-parabolic equations of the second order with non-divergent structure (see [7]), we have

$$\sup_{\tilde{C}^{11}} \tilde{u}_R \leq c_{3.16}(\gamma, n) \cdot \inf_{\tilde{C}^{13}} \tilde{u}_R \leq \tilde{u}_R(0, -b).$$

Returning to the variables (x,t),

$$\sup_{C^{11}} u \leq c_{3.16} \cdot u(0, -bR^2).$$

Now, to complete the proof of estimate (77), it is enough to apply Corollary 4.2.

The theorem is proved.

5. Conclusion

In this work, we have comprehensively investigated the **qualitative properties of solutions** for a class of **non-uniformly and strongly degenerate second-order elliptic-parabolic equations**. The presence of strong and non-uniform degeneracies introduces significant mathematical challenges, as the classical regularity theory and standard energy methods cannot be directly applied due to the loss of ellipticity or parabolicity in arbitrary subdomains.

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